Harmonic-Oscillator-Based Effective Theory

- Review: Bloch-Horowitz solutions for effective interactions and operators
- Connections with contact-gradient expansions
  - ◇ initial work with Luu on the running of the coefficients
  - ◇ re-examination of individual matrix elements – deeply bound vs. valence orbitals
- Harmonic oscillator-based effective theory
  - ◇ as expansion around q $\sim$ 1/b: removing operator mixing
  - ◇ T resummation and contract-gradient expansion
  - ◇ implications for potentials, b, halo nuclei...
- Fitting contact-gradient expansions to low-energy nuclear data

- Review: Bloch-Horowitz generates Hermitian but energy-dependent effective interactions and operators. We explore a bare H of the form

$$H = \frac{1}{2} \sum_{i,j=1}^{A} (T_{ij} + V_{ij})$$

where V represents a two-body potential like $av18$ and T is the two-body (relative) kinetic energy

$$H^{eff} = H + H \frac{1}{E - Q_{SM} H} Q_{SM} H$$

$$H^{eff} |\Psi_{SM}\rangle = E |\Psi_{SM}\rangle \quad |\Psi_{SM}\rangle = (1 - Q_{SM}) |\Psi\rangle$$

- Solved self-consistently: E is the exact eigenvalue
- $P_{SM} = 1 - Q_{SM}$ is defined by Λ_{SM} and b

- Λ_{SM}: retention of a complete set of $\Lambda_{SM}\ \hbar\omega$ excitations produces a separable space and a translation-invariant effective interaction
- Results completely independent of parameter choices if the effective theory is executed properly
- P-space wave function is the restriction of the exact wave function to P: wave function evolves simply
- Thus a nontrivial normalization that approaches 1 as $\Lambda_{SM} \rightarrow \infty$
- Calculations done both by explicitly summing over Q (140 $\hbar\omega$, D; 70 $\hbar\omega$, ^{3}He/^{3}H: C.-L. Song) and by a momentum-space integration over all excitations (T. Luu)
- "Test data" for examining effective interaction, operator behavior

- Evolution of ^{3}He $av18$ SM wave function with Λ_{SM}

state	amplitude					
	$0\hbar\omega$	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\ \hbar\omega$	exact
	(31.1%)	(57.4%)	(70.0%)	(79.8%)	(85.5%)	(100%)
$\|0,1\rangle$	0.5579	0.5579	0.5579	0.5579	0.5579	0.5579
$\|2,1\rangle$	0.0000	0.0463	0.0461	0.0462	0.0462	0.0463
$\|2,2\rangle$	0.0000	-0.4825	-0.4824	-0.4824	-0.4824	-0.4826
$\|2,3\rangle$	0.0000	0.0073	0.0073	0.0073	0.0073	0.0073
$\|4,1\rangle$	0.0000	0.0000	-0.0204	-0.0204	-0.0204	-0.0205
$\|4,2\rangle$	0.0000	0.0000	0.1127	0.1127	0.1127	0.1129
$\|4,3\rangle$	0.0000	0.0000	-0.0419	-0.0420	-0.0421	-0.0423

- Evolution of effective interaction m.e.s with Λ_{SM}

	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\ \hbar\omega$
$\langle 0,1 \mid H^{eff} \mid 2,1 \rangle$	-4.874	-3.165	-0.449	1.279
$\langle 0,1 \mid H^{eff} \mid 2,5 \rangle$	-0.897	-1.590	-1.893	-2.208
$\langle 2,1 \mid H^{eff} \mid 2,2 \rangle$	6.548	-2.534	-4.144	-5.060

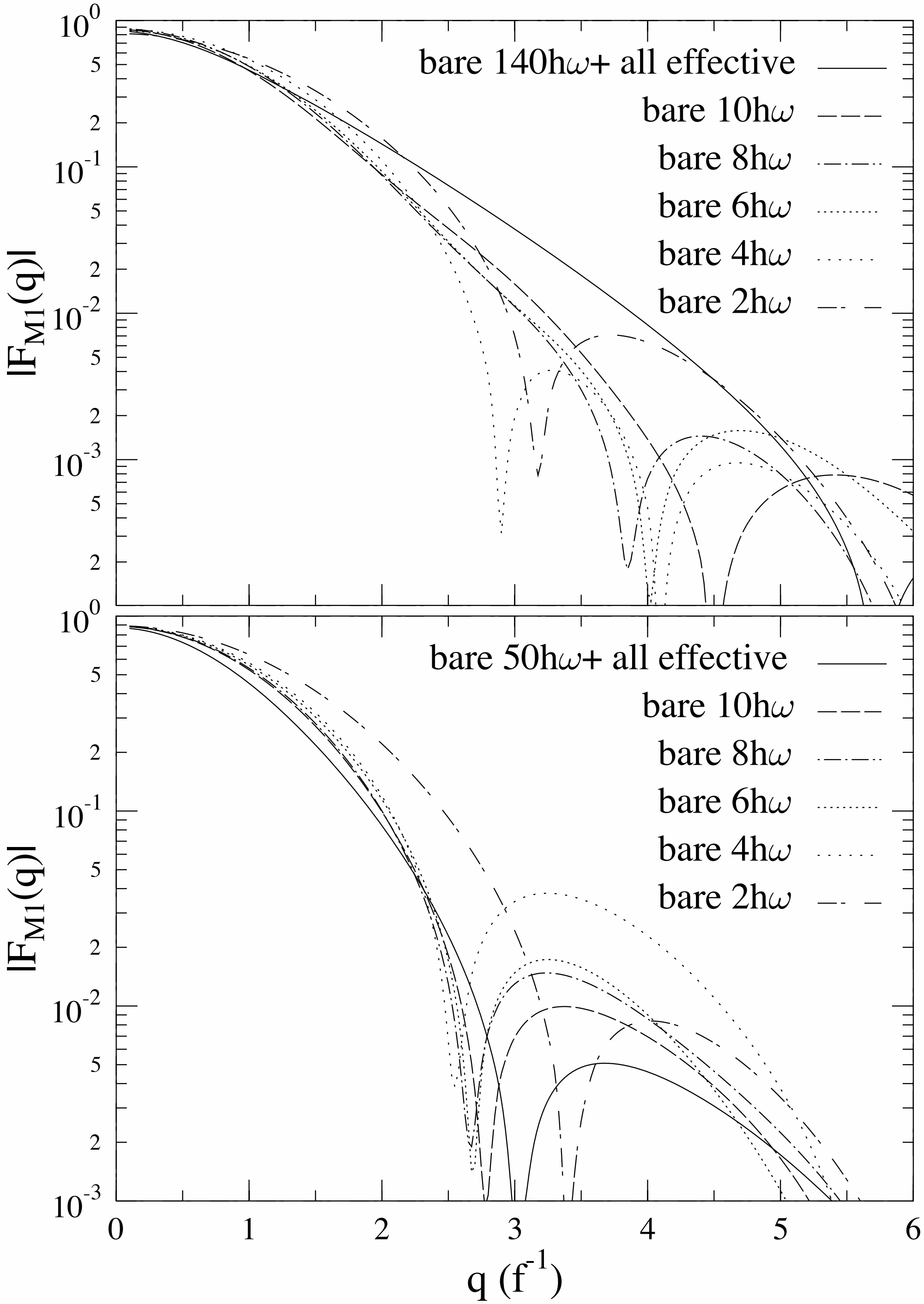
- Evolution of observables: ground-state energies do not change even with large changes in b (to accuracy of 1 or 10 keV, for d or ^{3}He/^{3}H)
- Similarly, effective operators are defined

$$\hat{O}^{eff} = (1 + HQ\frac{1}{E_f - HQ})\hat{O}(1 + \frac{1}{E_i - QH}QH)$$

and must be evaluated between "SM" wave functions properly normed

$$1 = \langle\Psi_i|\Psi_i\rangle = \langle\Psi_i^{`SM}|\hat{1}^{eff}|\Psi_i^{SM}\rangle \tag{1}$$

(also for $|\Psi_f^{SM}\rangle$)



bare 140hω+ all effective
bare 10hω
bare 8hω
bare 6hω
bare 4hω
bare 2hω
bare 50hω+ all effective
bare 10hω
bare 8hω
bare 6hω
bare 4hω
bare 2hω
|F_M1(q)|
|F_M1(q)|
q (f^-1)

- This work was intended as a check against a direct ET treatment of interactions and operators (our goal): began to look at this in 1999 (WH and Luu)
- Wrote down the most general nonlocal interactions of the contact-gradient form, e.g., the s-wave momentum expansion

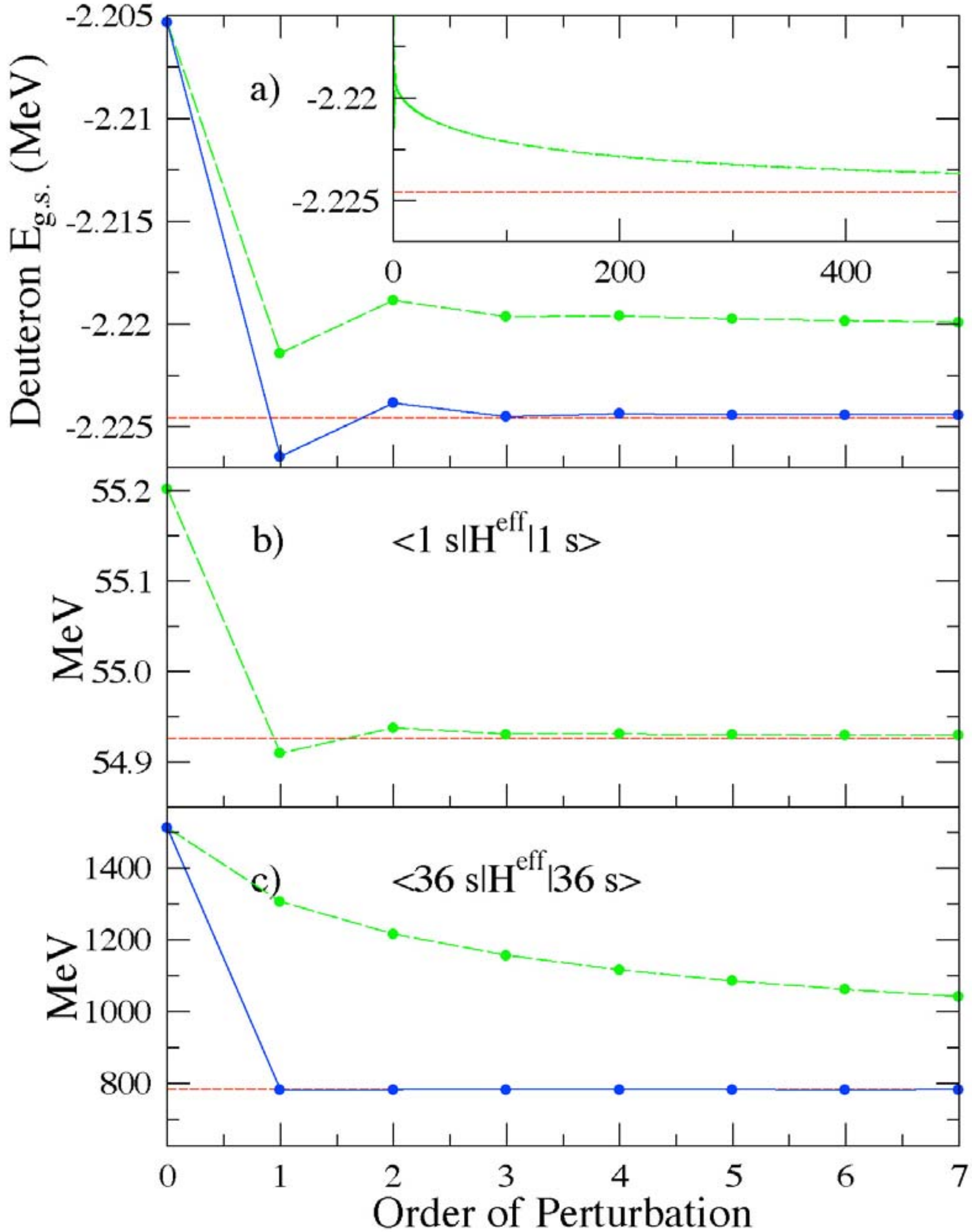
$$\mathrm{LO}: \quad a^{ss}_{LO}\delta(\mathbf{r})$$

$$\mathrm{NLO}: \quad a^{ss}_{NLO}(\Lambda_{SM},b)(\overleftarrow{\nabla}^2\delta(\mathbf{r})+\delta(\mathbf{r})\overrightarrow{\nabla}^2)$$

$$\mathrm{NNLO}: \quad a^{ss,22}_{NNLO}(\Lambda_{SM},b)\overleftarrow{\nabla}^2\delta(\mathbf{r})\overrightarrow{\nabla}^2+a^{ss,40}_{NNLO}(\Lambda_{SM},b)(\overleftarrow{\nabla}^4\delta(\mathbf{r})+\delta(\mathbf{r})\overrightarrow{\nabla}^4)$$

- Encountered odd running of couplings, associated with nonperturbative effects of T

- Q_{SM} defined by $\Lambda_{SM}\hbar\omega$ (translational invariance)
- This is an energy cut, not a momentum cut
- $\langle q \rangle_{1s} \sim 1/b$: expansion about an intermediate scale
- Combinations of high-energy configurations can be soft
- The competition between V and T depends on the nuclear binding energy relative to the first open channel, typically $\sim$ 10 MeV – a sharp variation not represented in HO SM
- This physics is generally many-body
- Luu and WH studied, initially, the non-perturbative long-range wave function
- H.O. has long- and short-range problems, plane-wave contact-gradient expansion can account for only the later



Deuteron $E_{g.s.}$ (MeV)
-2.205
-2.21
-2.215
-2.22
-2.225
a)
-2.22
-2.225
0
200
400
55.2
55.1
55.0
54.9
MeV
b)
<1 s|H^{eff}|1 s>
1400
1200
1000
800
MeV
c)
<36 s|H^{eff}|36 s>
0
1
2
3
4
5
6
7
Order of Perturbation

- Re-sum QT to all orders in $H_{eff} = H + H\frac{1}{E-QH}QH$

$$\begin{array}{rcl} \text{edge states} & \leftrightarrow & \text{deep states} \\ \langle\alpha|T + TQ\frac{1}{E-QT}QT|\beta\rangle + & \leftrightarrow & \langle\alpha|T|\beta\rangle + \\ \langle\alpha|\frac{E}{E-TQ}V\frac{E}{E-QT}|\beta\rangle + & \leftrightarrow & \langle\alpha|V|\beta\rangle + \\ \langle\alpha|\frac{E}{E-TQ}V\frac{1}{E-QH}QV\frac{E}{E-QT}|\beta\rangle & \leftrightarrow & \langle\alpha|V\frac{1}{E-QH}QV|\beta\rangle \end{array}$$

- Deep states $\sim$ plane-wave states: $P \leftrightarrow Q$ uncoupled by T
- Edge states maximally couple: T ladder operator
- QT summation $\rightarrow$ local operators acts on external legs
- CM-preserving new operators, but suggestive of a basis transformation too

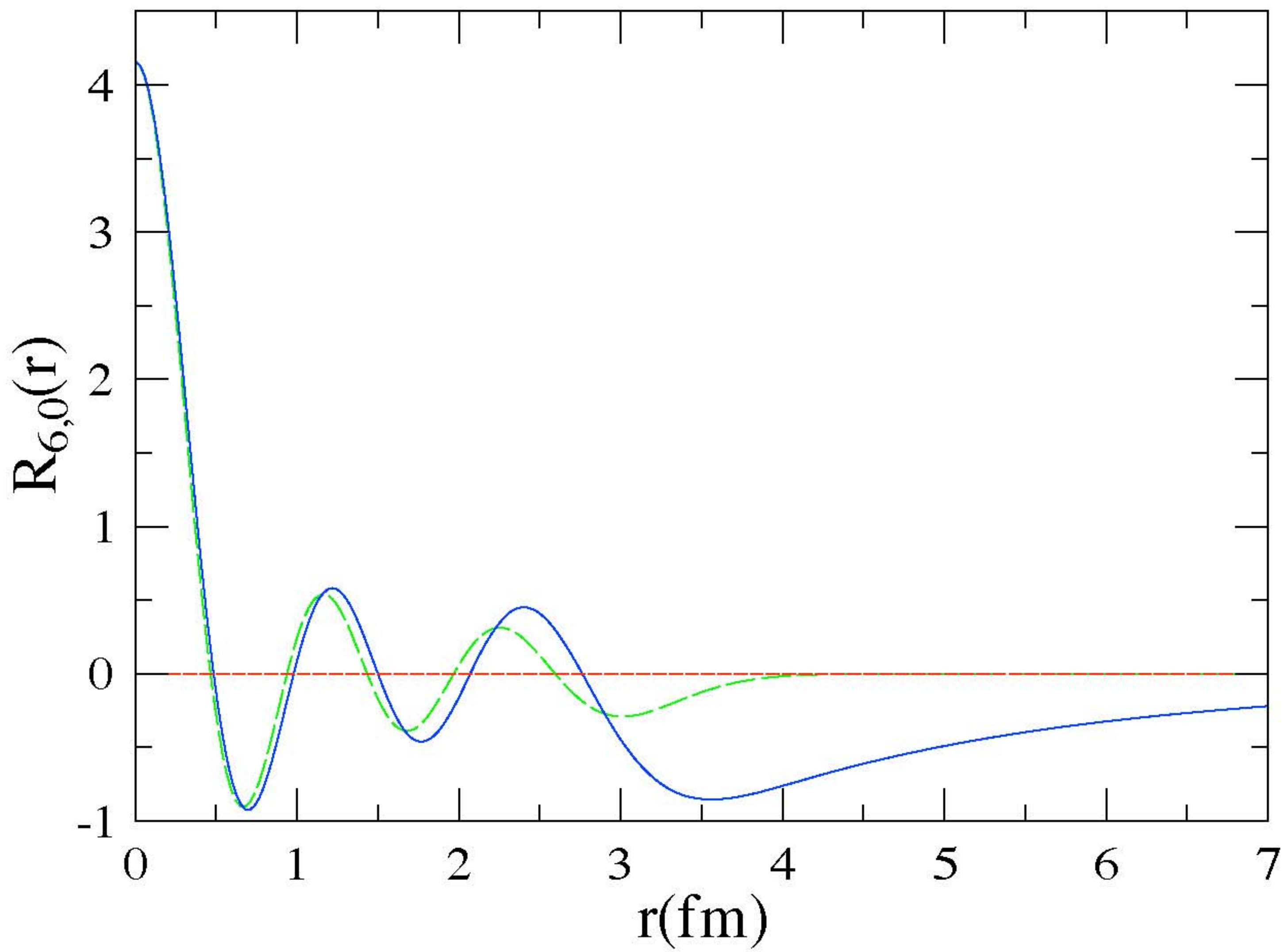
- That is, this can be rewritten

$$\langle \alpha | H^{eff} | \beta \rangle = \langle \alpha | T | \tilde{\beta} \rangle + \langle \tilde{\alpha} | V + V \frac{1}{E - QH} QV | \tilde{\beta} \rangle$$

where

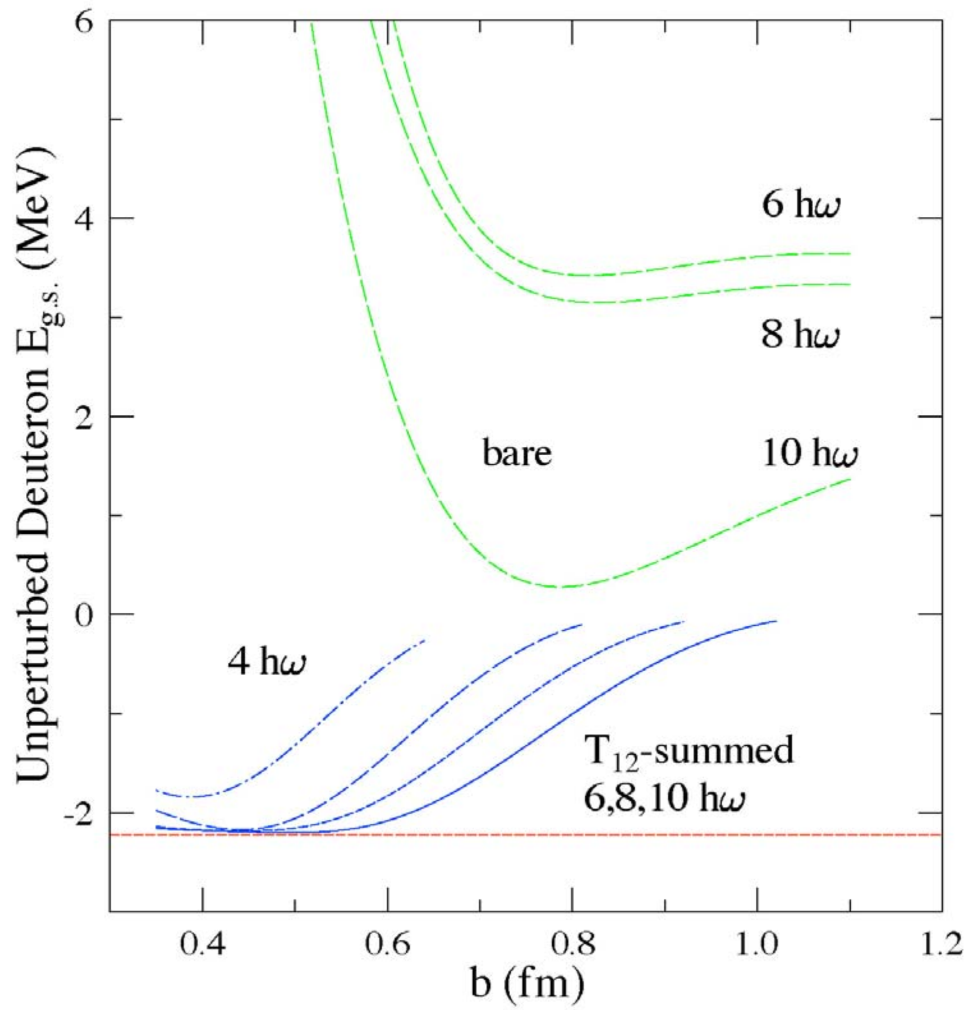
$$|\tilde{\alpha}\rangle = \frac{E}{E - QT} |\alpha\rangle$$

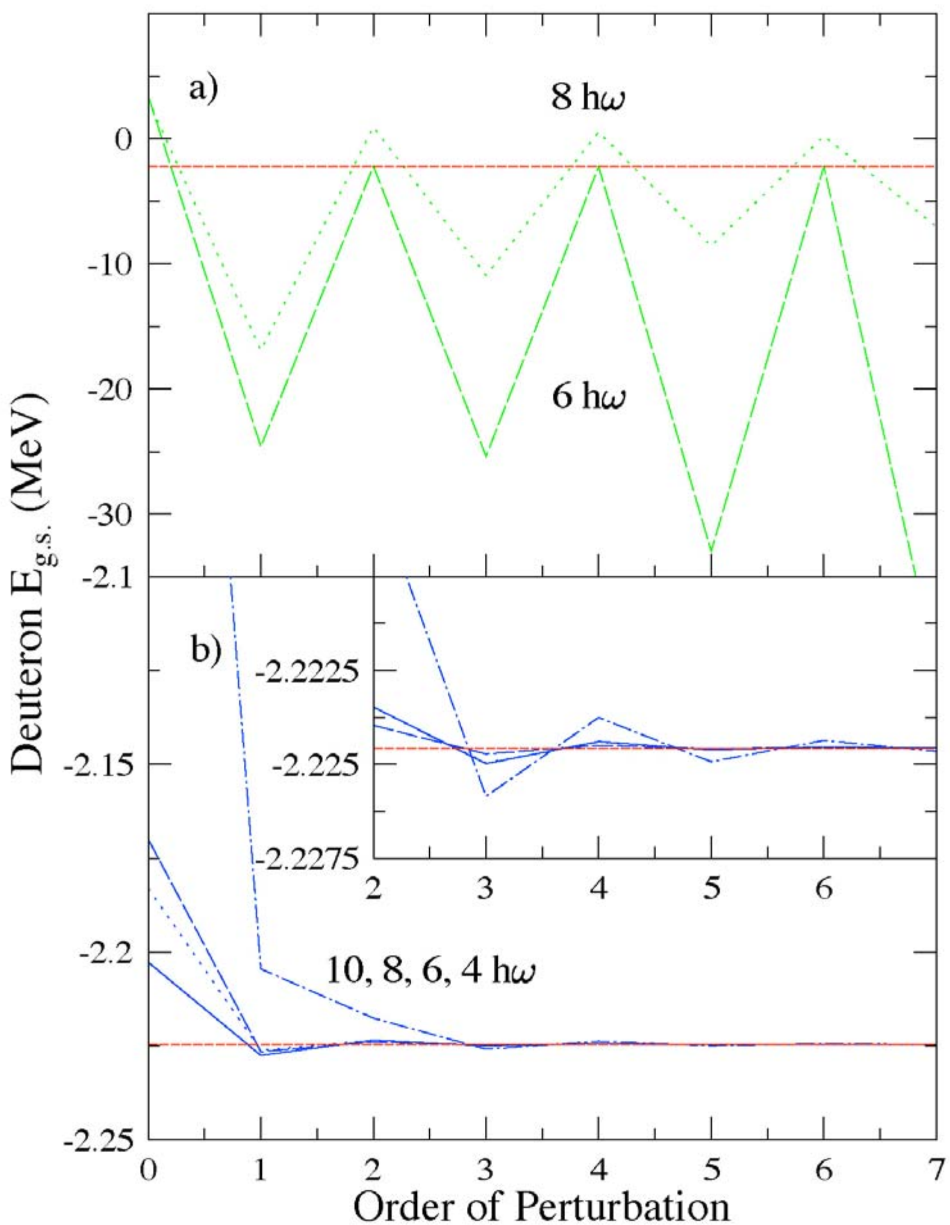
- For all nonedge states, $|\tilde{\alpha}\rangle = |\alpha\rangle$



$R_{6,0}(r)$
r(fm)
4
3
2
1
0
-1
0
1
2
3
4
5
6
7

- Deuterium g.s. convergence with a bare interaction!



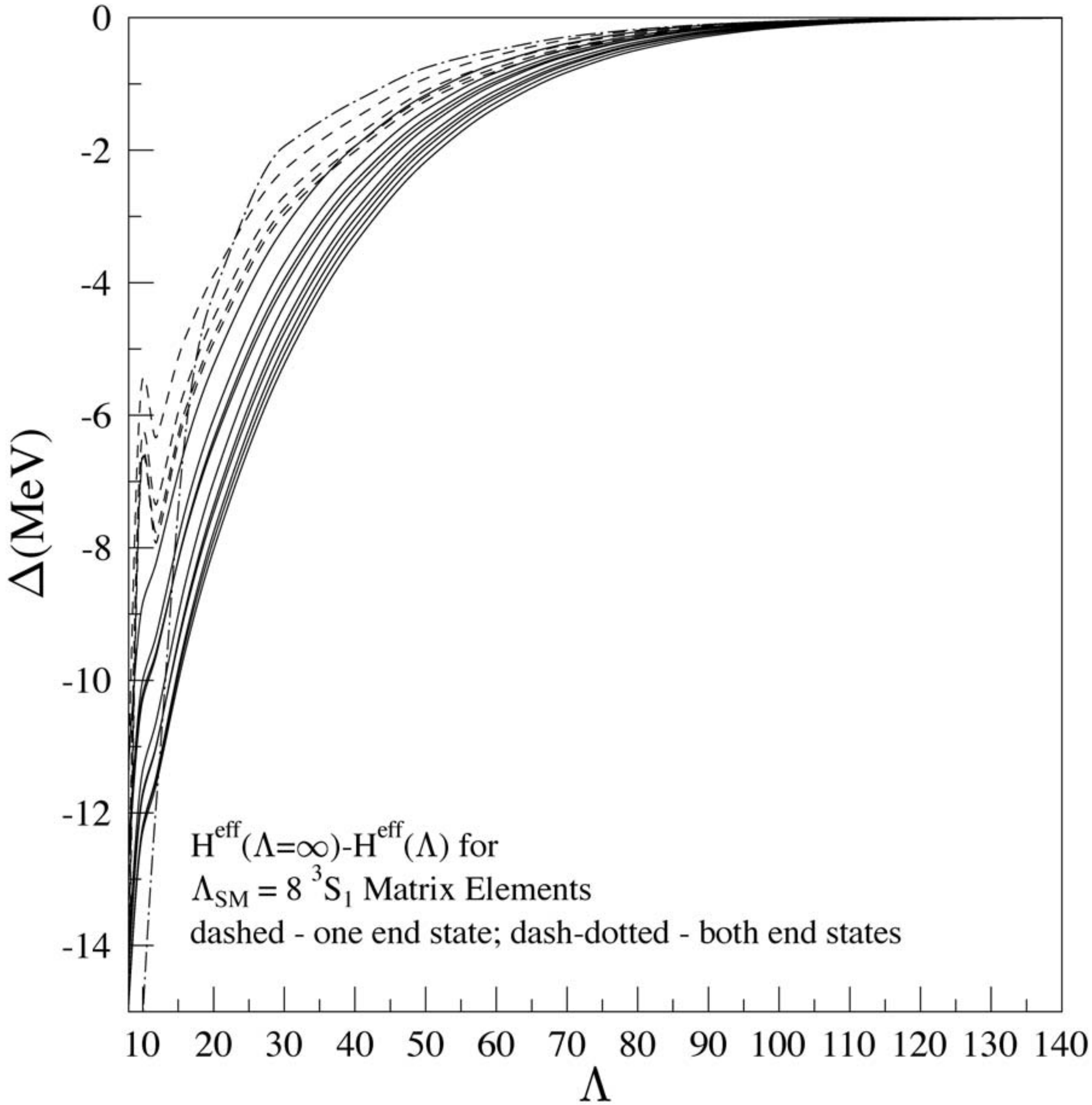


a)
8 hω
6 hω
b)
10, 8, 6, 4 hω
Deuteron $E_{g.s.}$ (MeV)
Order of Perturbation

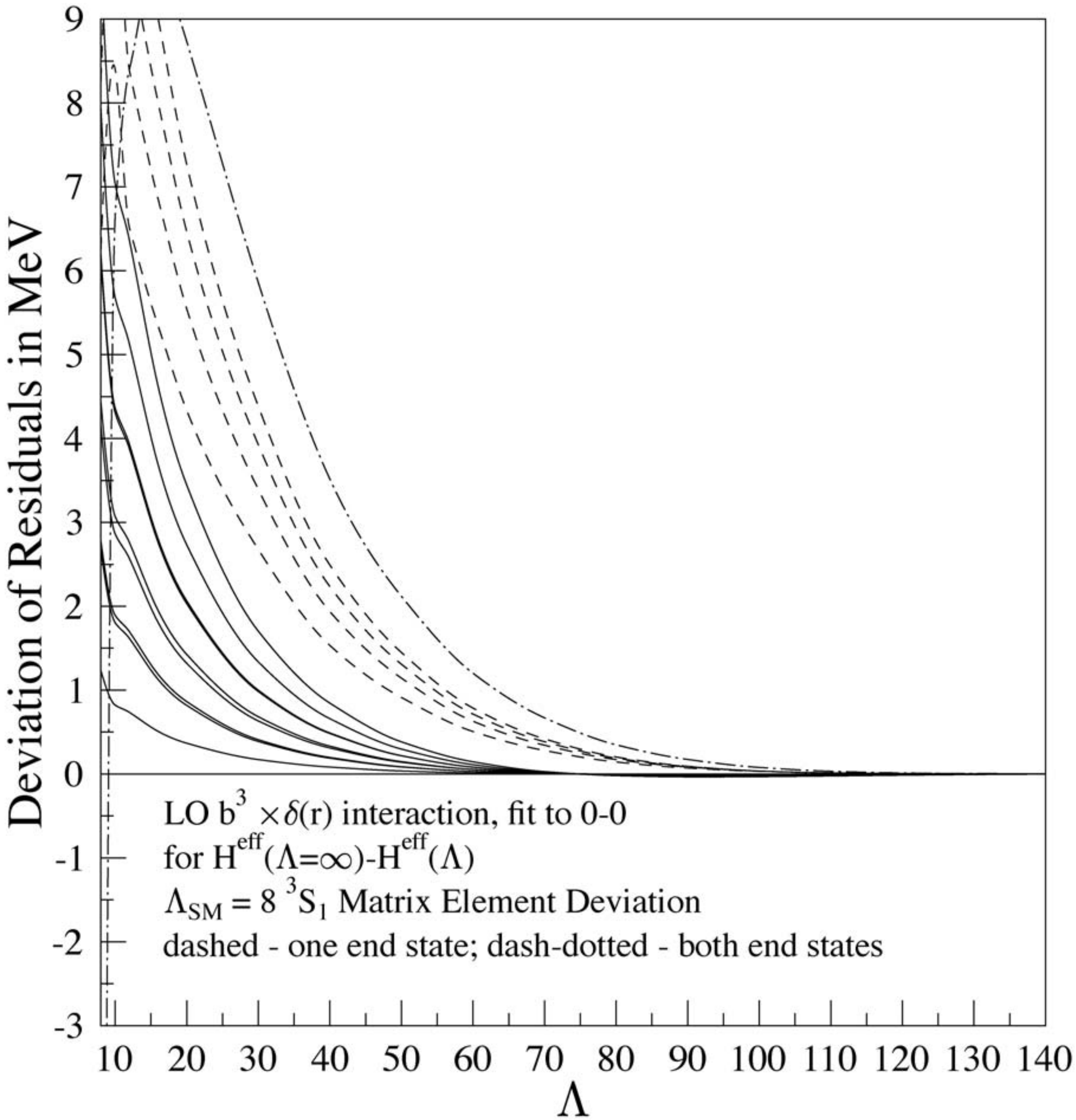
- The general case where two-body length scale not connected with nuclear size explored in A=3
- Q-space interaction decomposed into iterated two-body Fadeev bubbles (two-body ladders)
- these summed in momentum space to all orders in V
- QT summation again carried out in closed form to all orders, forming the three-body $|\tilde{\alpha}\rangle$
- H_{eff} again converged as a perturbation in two-body ladders (even though published work did not do this optimally)
- If this works in three-body case, should work in general (if done properly)
- Contradicts old lore from early 1970s

HOBET revisited: the old problem illustrated more clearly

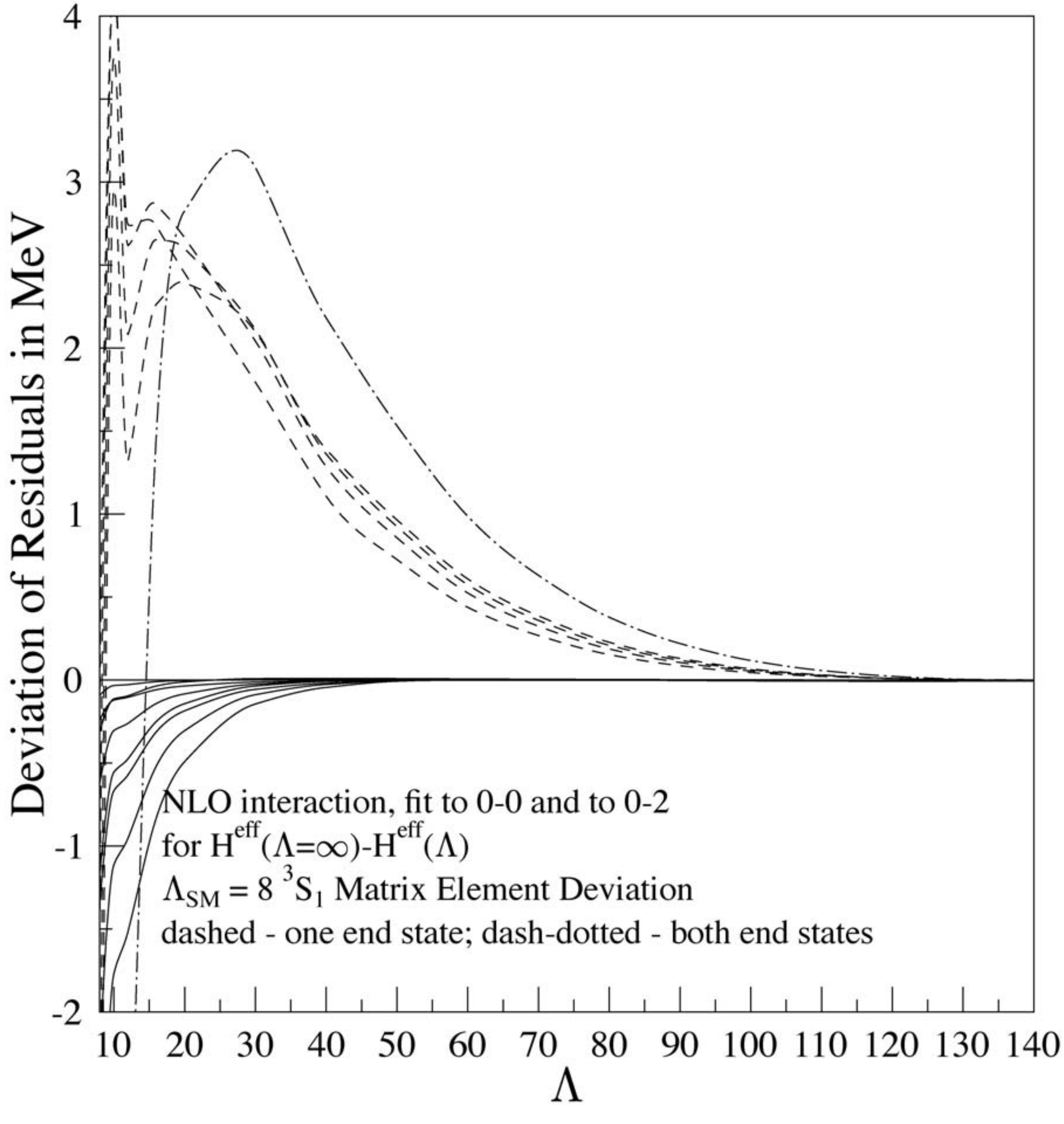
- HOBET: are there simple, accurate contact-gradient expansions in HO-based effective theory?
- If so, what is their structure, how can they be determined?
- Edge -deep states $\Rightarrow$ compare behavior of $\langle\alpha|H_{eff}|\beta\rangle$
  - ◇ explore simple example, deuteron with Λ_{SM} = 8
  - ◇ move down in Q-space from infinity to $\Lambda \geq \Lambda_{SM}$ in steps
  - ◇ represent physics above Λ by $a_{LO},\ a_{NLO},\ a_{NNLO}...$, fit to $av18$ H_{eff} m.e.s for deep(est) states
  - ◇ do LO, NLO, NNLO interactions improve systematically?
  - ◇ on reaching $\Lambda = \Lambda_{SM}$, does an accurate $H + H_{eff}^{NNLO}$ exist?



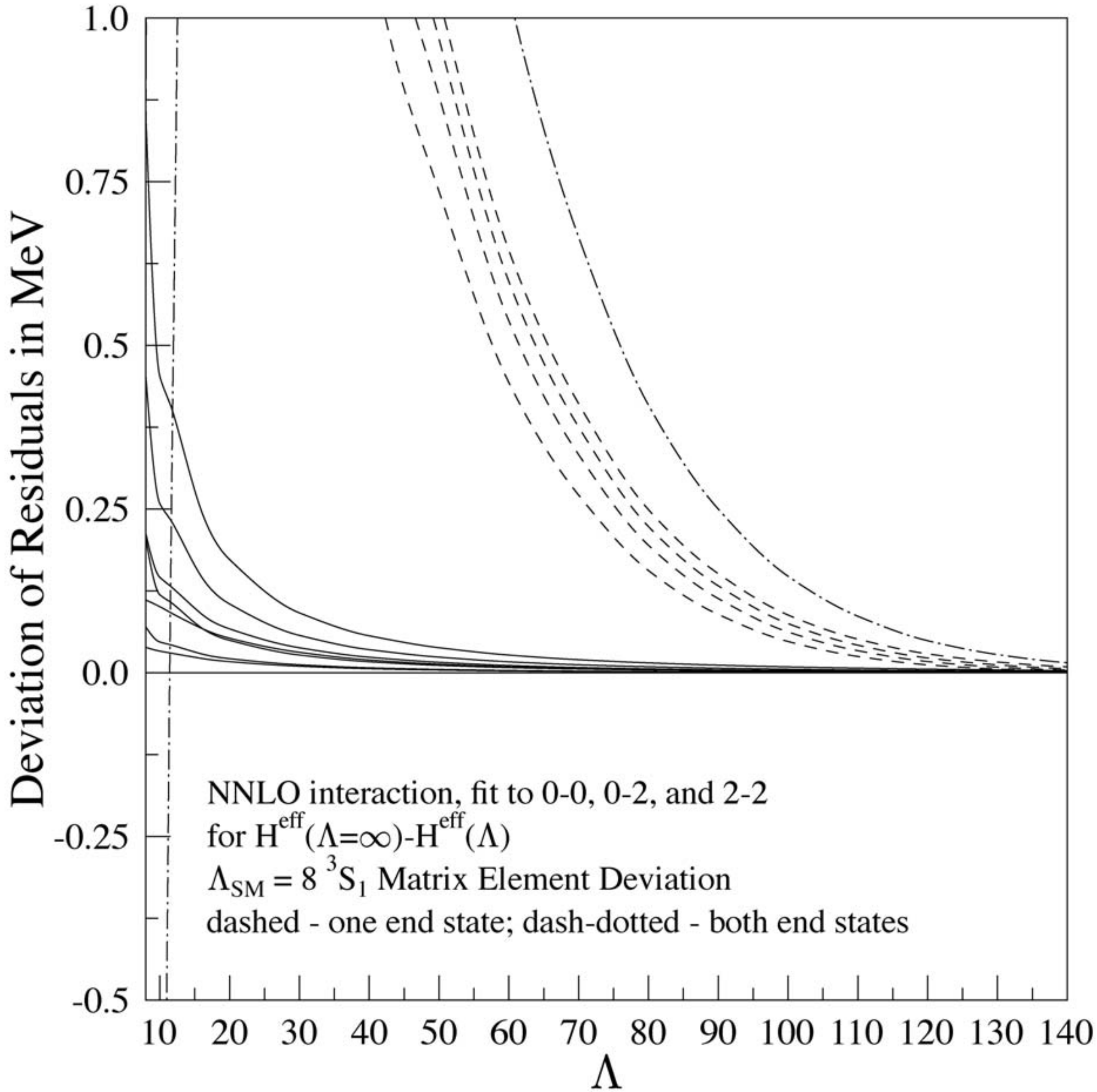
0
-2
-4
-6
-8
-10
-12
-14
Δ(MeV)
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
$H^{eff}(\Lambda=\infty)-H^{eff}(\Lambda)$ for
Λ_{SM} = 8 $^3S_1$ Matrix Elements
dashed - one end state; dash-dotted - both end states



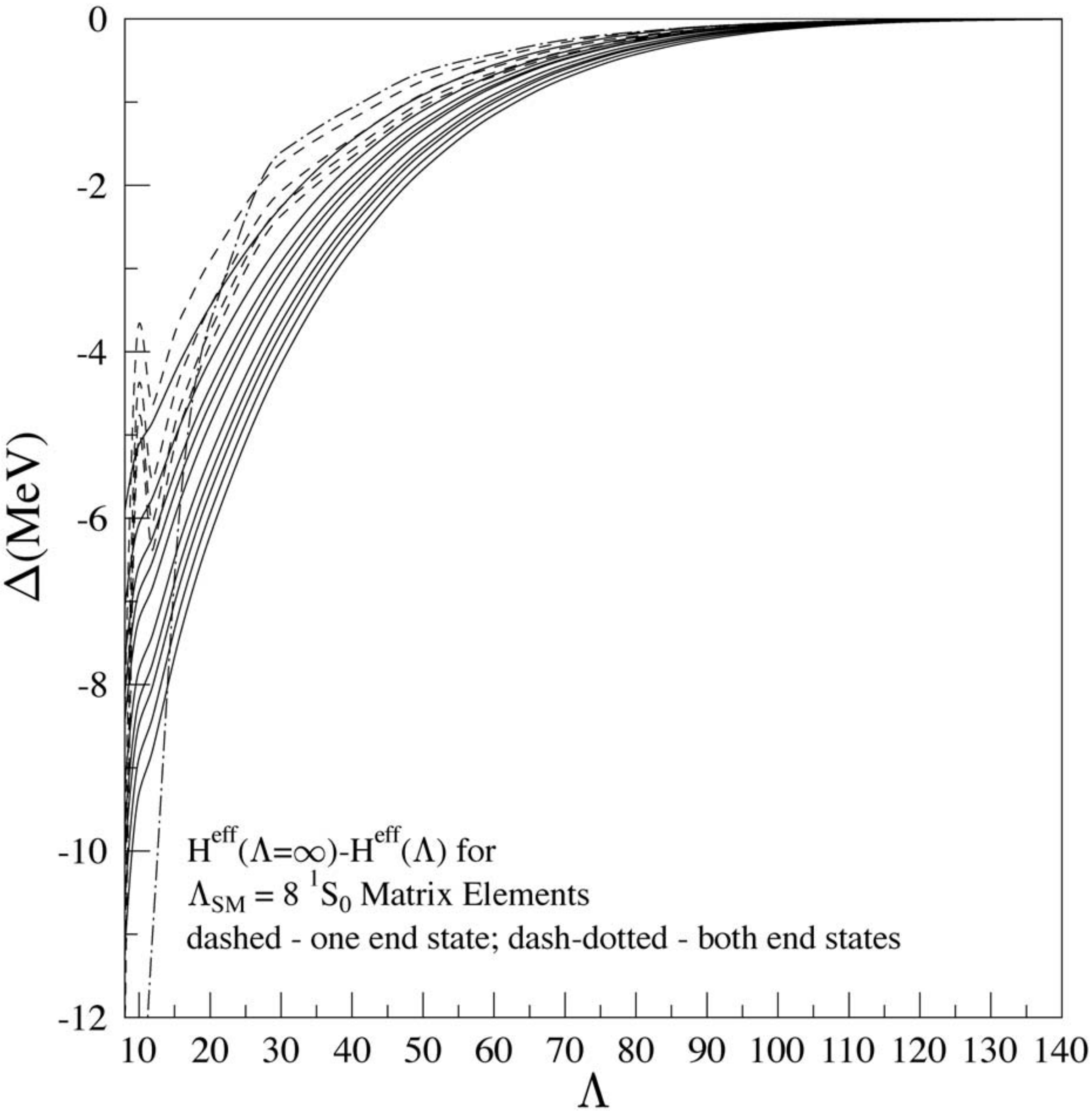
Deviation of Residuals in MeV
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
LO b^3 ×δ(r) interaction, fit to 0-0
for H^{eff}(Λ=∞)-H^{eff}(Λ)
Λ$_{SM}$ = 8 ^{3}S$_1$ Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



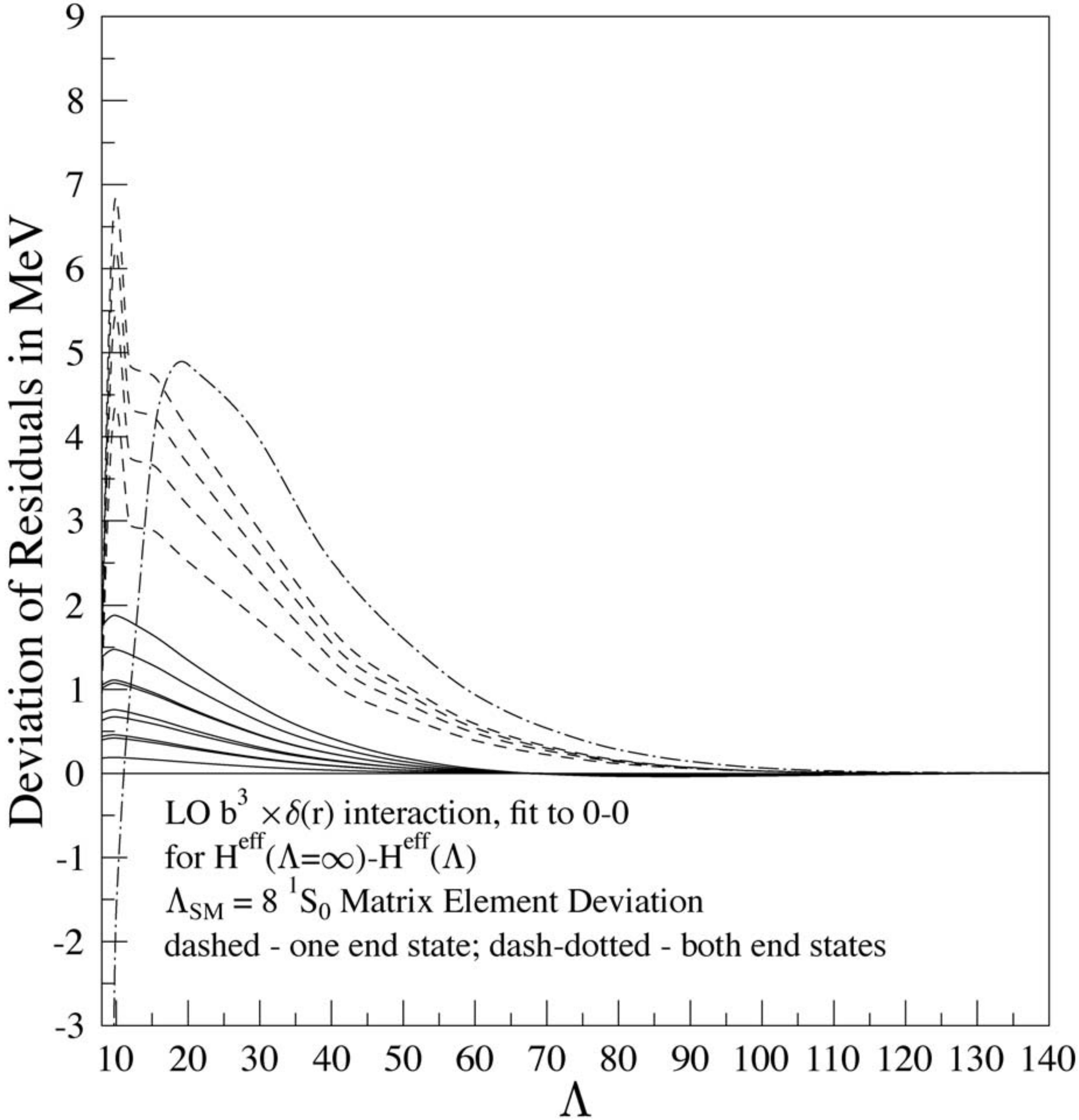
NLO interaction, fit to 0-0 and to 0-2
for $H^{eff}(\Lambda=\infty)$-$H^{eff}(\Lambda)$
Λ_{SM} = 8 $^3S_1$ Matrix Element Deviation
dashed - one end state; dash-dotted - both end states
Deviation of Residuals in MeV
Λ
4
3
2
1
0
-1
-2
10
20
30
40
50
60
70
80
90
100
110
120
130
140



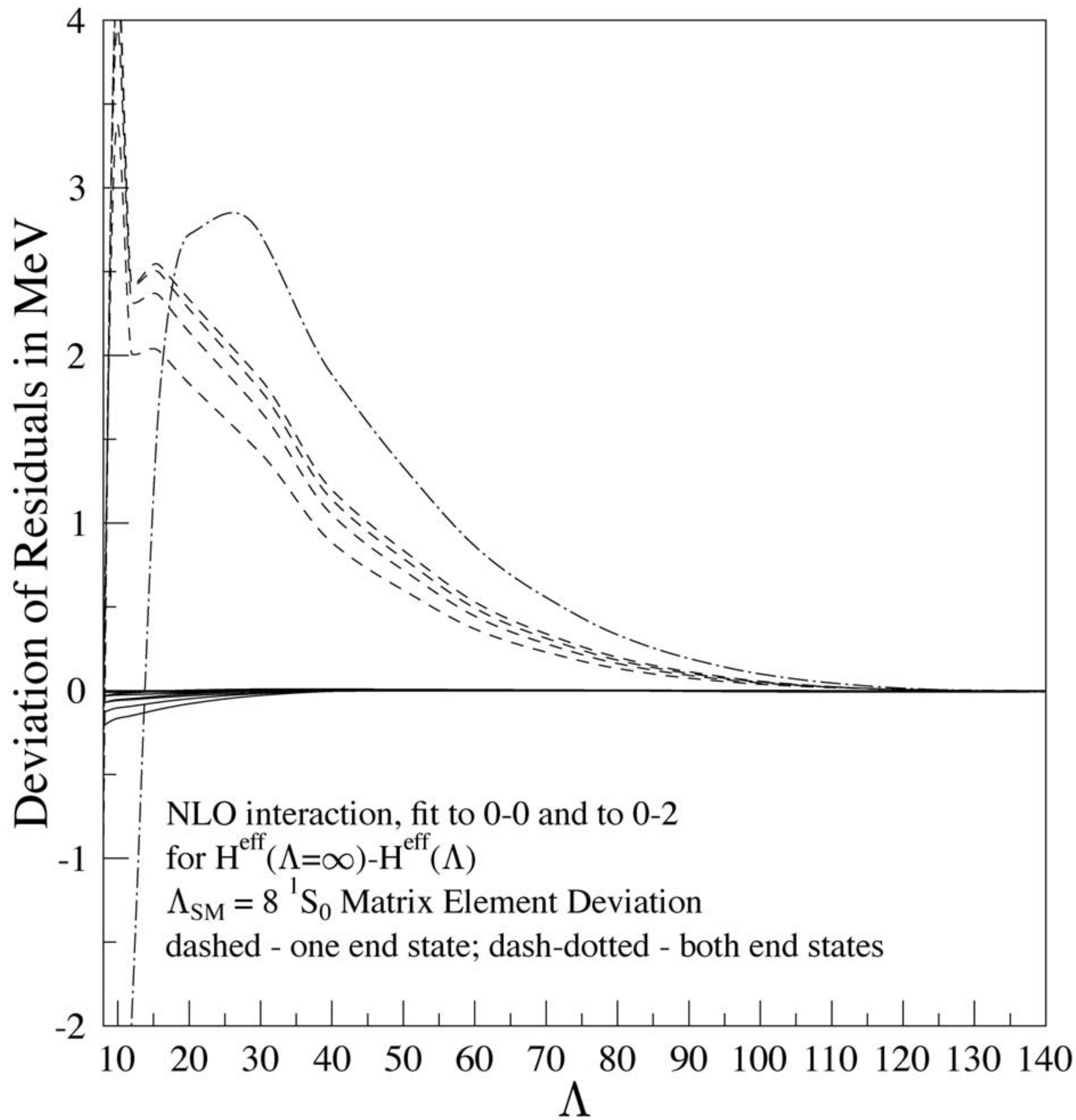
Deviation of Residuals in MeV
1.0
0.75
0.5
0.25
0.0
-0.25
-0.5
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
NNLO interaction, fit to 0-0, 0-2, and 2-2
for $H^{eff}(\Lambda=\infty)-H^{eff}(\Lambda)$
$\Lambda_{SM} = 8$ $^3S_1$ Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



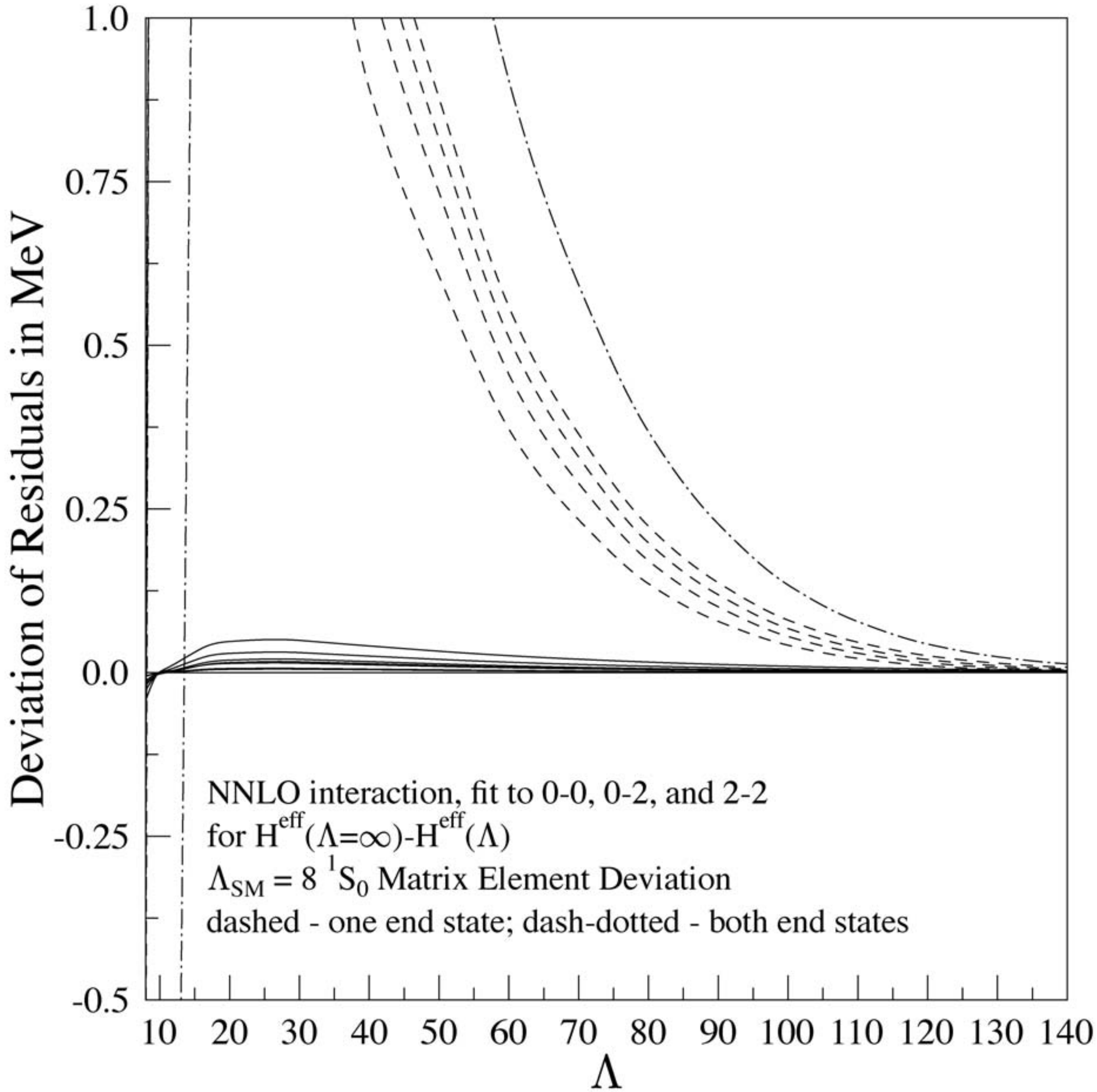
0
-2
-4
-6
-8
-10
-12
Δ(MeV)
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
H^eff(Λ=∞)-H^eff(Λ) for
Λ_SM = 8 ^1S_0 Matrix Elements
dashed - one end state; dash-dotted - both end states



Deviation of Residuals in MeV
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
LO b^3 × δ(r) interaction, fit to 0-0
for H^eff(Λ=∞)-H^eff(Λ)
Λ_SM = 8 ^1S_0 Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



Deviation of Residuals in MeV
4
3
2
1
0
-1
-2
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
NLO interaction, fit to 0-0 and to 0-2
for $H^{eff}(\Lambda=\infty)$-$H^{eff}(\Lambda)$
Λ_{SM} = 8 $^1S_0$ Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



1.0
0.75
0.5
0.25
0.0
-0.25
-0.5
Deviation of Residuals in MeV
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
NNLO interaction, fit to 0-0, 0-2, and 2-2
for $H^{eff}(\Lambda=\infty)-H^{eff}(\Lambda)$
Λ_{SM} = 8 $^1S_0$ Matrix Element Deviation
dashed - one end state; dash-dotted - both end states

Step #1 in HOBET formulation: recast as expansion in $1/b$

- Standard EFT approaches are expansions around $\vec{k}=0$

$$\overrightarrow{\nabla}^n \exp i\vec{k}\cdot\vec{r} = 0, n = 1,2,....$$

- By analogy demand in HOBET

$$\overrightarrow{\nabla}^n \psi_{1s}(b) = 0, n = 1,2,...$$

- These leads to the HOBET form of EFT operators, e.g.,

$$a^{ss}_{LO}(\Lambda,b)e^{r^2/2}\delta(\mathbf{r})e^{r^2/2}$$

$$a^{ss}_{NLO}(\Lambda,b)e^{r^2/2}(\overleftarrow{\nabla}^2\delta(\mathbf{r})+\delta(\mathbf{r})\overrightarrow{\nabla}^2)e^{r^2/2}$$

$$a^{ss,22}_{NNLO}(\Lambda,b)e^{r^2/2}\overleftarrow{\nabla}^2\delta(\mathbf{r})\overrightarrow{\nabla}^2 e^{r^2/2}+a^{ss,40}_{NNLO}e^{r^2/2}(\Lambda,b)(\overleftarrow{\nabla}^4\delta(\mathbf{r})+\delta(\mathbf{r})\overrightarrow{\nabla}^4)e^{r^2/2}$$

- Acts on polynomials $\leftrightarrow$ short-range behavior
- Removes all operator mixing: e.g., a_{LO} fixed in LO to $n=1 \leftrightarrow n=1$, not affected by higher orders
- The expansion is in nodal quantum numbers, e.g.,

$$\vec{\nabla}^2 \sim (n-1) \qquad \vec{\nabla}^4 \sim (n-1)(n-2)$$

so that matrix elements become trivial to evaluate to any order

- Leading order in n contribution agrees with plane-wave result (plane wave results recovered as $b \rightarrow \infty$)
- Operator coefficients are a generalization of Talmi integrals

$$e.g., a_{NNLO}^{ss,22} \sim \int_0^\infty \int_0^\infty e^{-r_1^2} r_1^2 V(r_1,r_2) r_2^2 e^{-r_2^2} r_1^2 r_2^2 dr_1 dr_2$$

Step #2: resum QT and evaluate consequences for interaction

- Recall that

$$
\begin{array}{rcl}
\text{edge states} & \leftrightarrow & \text{deep states} \\
\langle\alpha|T + TQ\dfrac{1}{E-QT}QT|\beta\rangle + & \leftrightarrow & \langle\alpha|T|\beta\rangle + \\
\langle\alpha|\dfrac{E}{E-TQ}V\dfrac{E}{E-QT}|\beta\rangle + & \leftrightarrow & \langle\alpha|V|\beta\rangle + \\
\langle\alpha|\dfrac{E}{E-TQ}V\dfrac{1}{E-QH}QV\dfrac{E}{E-QT}|\beta\rangle & \leftrightarrow & \langle\alpha|V\dfrac{1}{E-QH}QV|\beta\rangle
\end{array}
$$

- Summations over QT easily performed: raising/lowering operator
- Leads to a series of continued fractions $\tilde{g}_i(2E/\hbar\omega,\{\alpha_i\},\{\beta_i\})$, where $\alpha_i = (2n+2i+l-1/2)/2$, $\beta_i = \sqrt{(n+i)(n+i+l+1/2)}/2$

- For any operator O (O = V, $V\frac{1}{E-QV}QV$, etc.)

$$\langle n'l'|\frac{E}{E-TQ}O\frac{E}{E-QT}|nl\rangle = \sum_{i,j=0}\tilde{g}_j(n',l')\tilde{g}_i(n,l)\langle n'+j\ l|O|n+i\ l\rangle$$

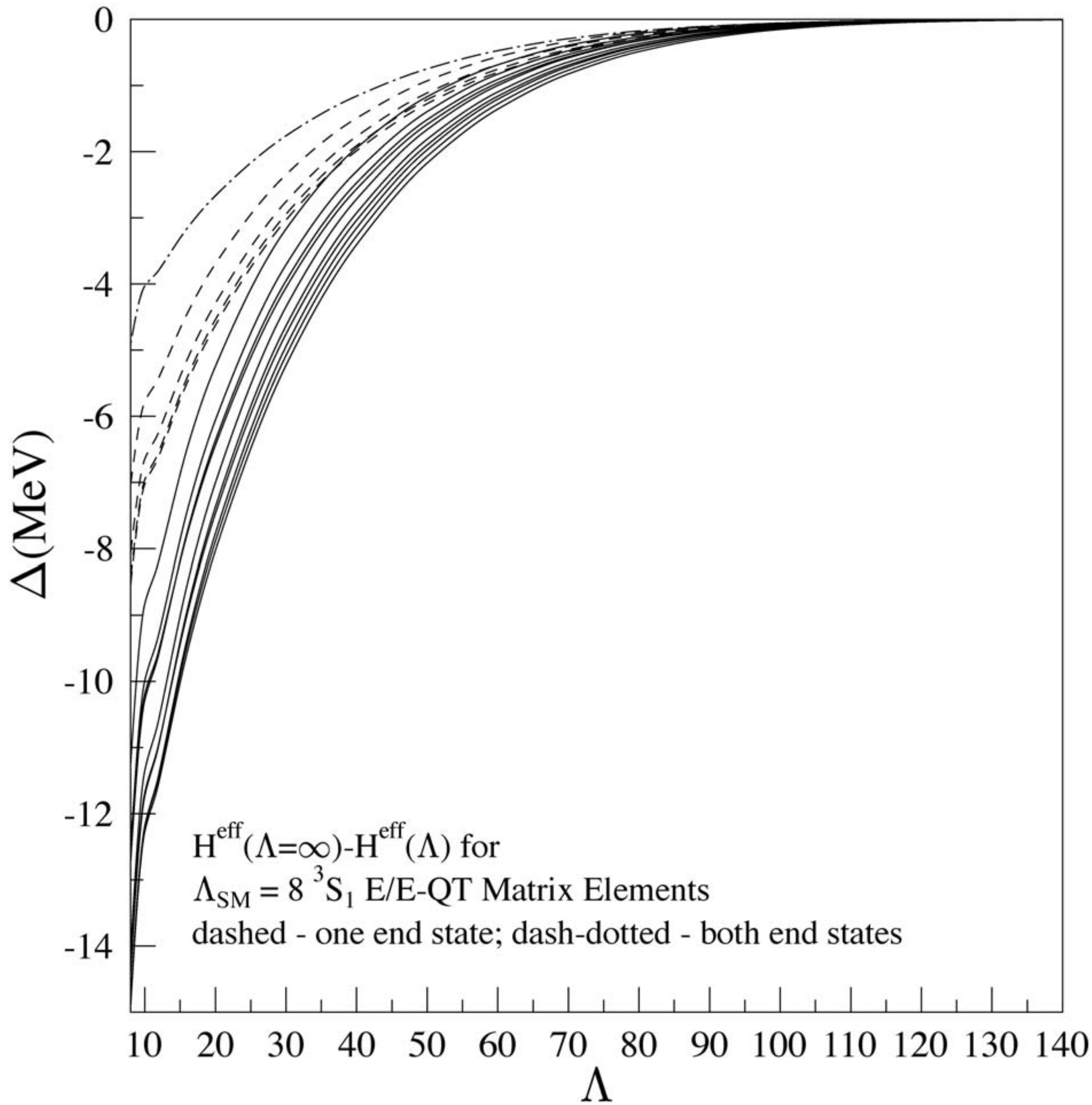
- Thus if $VGV \leftrightarrow a_{NL}, a_{NLO}, ...$, one finds an analytic renormalization governed by $E/\hbar\omega$, e.g.,

$$a_{LO} \rightarrow a'_{LO} = a_{LO} \times \sum_{i,j=0}\tilde{g}_j(n',l')\tilde{g}_i(n,l)$$

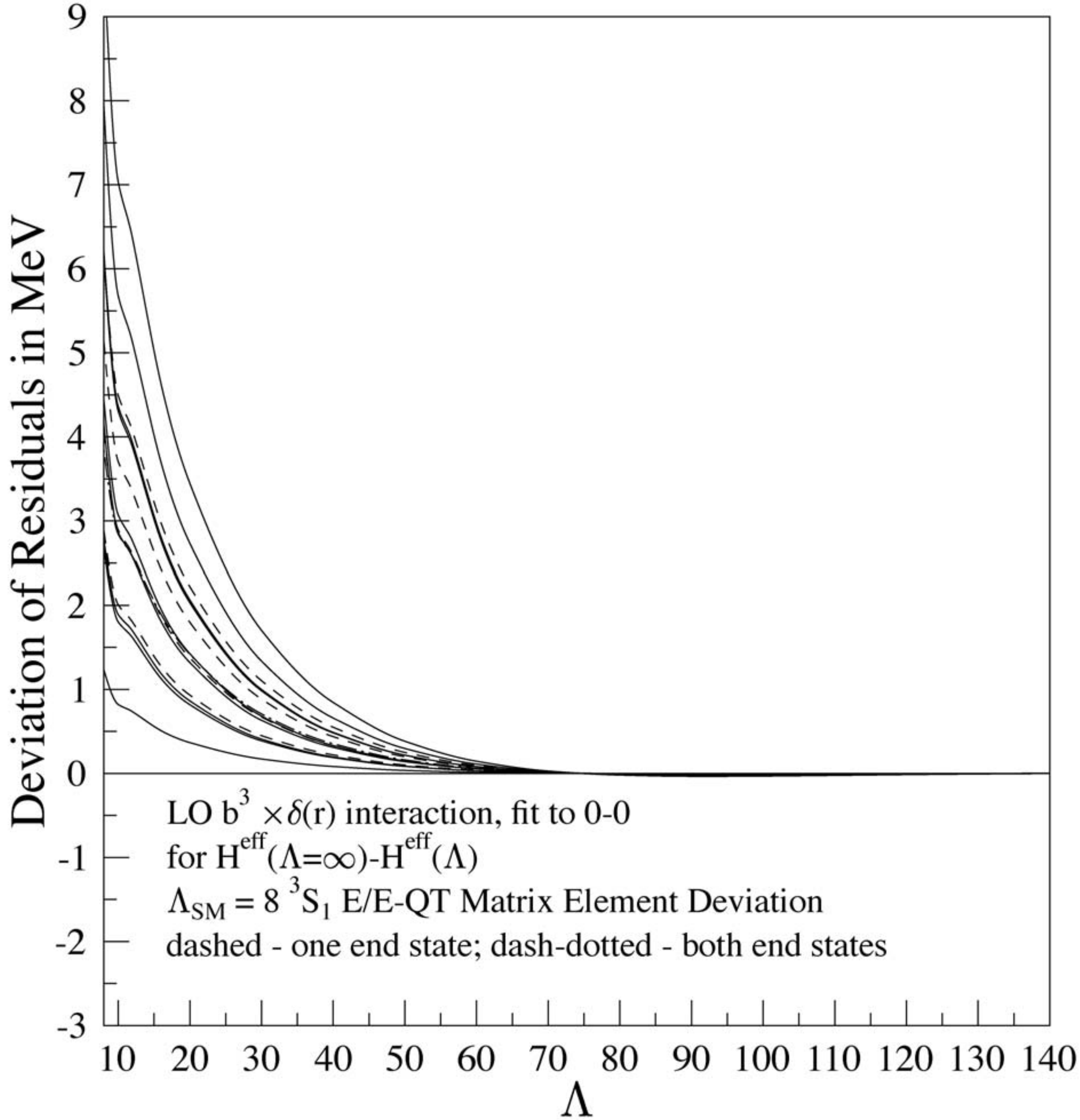
$$\left[\frac{\Gamma(n'+j+1/2)\Gamma(n+i+1/2)}{\Gamma(n'+1/2)\Gamma(n+1/2)}\right]^{1/2}\left[\frac{(n'-1)!(n-1)!}{(n'+j-1)!(n+i-1)!}\right]^{1/2}$$

- No new parameters have been introduced
- Can be generalized for A=3,4,5,...

- This is a general result for the shell model, a consequence of the strong $P-Q$ coupling driven by QT
- Plane-wave (e.g., Kuo-Brown g-matrix, V-low-k): T diagonal, so $VGV \leftrightarrow$ deep states: similar renormalization required
- Very physical: in extreme-halo-nucleus limit, a correct HOBET allows the valence nucleon to decouple from V
- Isolates and evaluates the $entire$ Bloch-Horowitz energy dependence has been identified: $VGV \sim$ energy-independent
- With $E_{gs} \sim$ few-10 MeV, very sensitive to excited-state energies: $a'_{LO}/a_{LO} \sim 0.25-0.50$ at 2.22 MeV
- In Lee-Suzuki beyond A=3?
  - ◇ if may be that this explains the “drifting” of b in no-core shell model



0
-2
-4
-6
-8
-10
-12
-14
Δ(MeV)
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
$H^{eff}(\Lambda=\infty)-H^{eff}(\Lambda)$ for
Λ_{SM} = 8 $^3S_1$ E/E-QT Matrix Elements
dashed - one end state; dash-dotted - both end states



Deviation of Residuals in MeV
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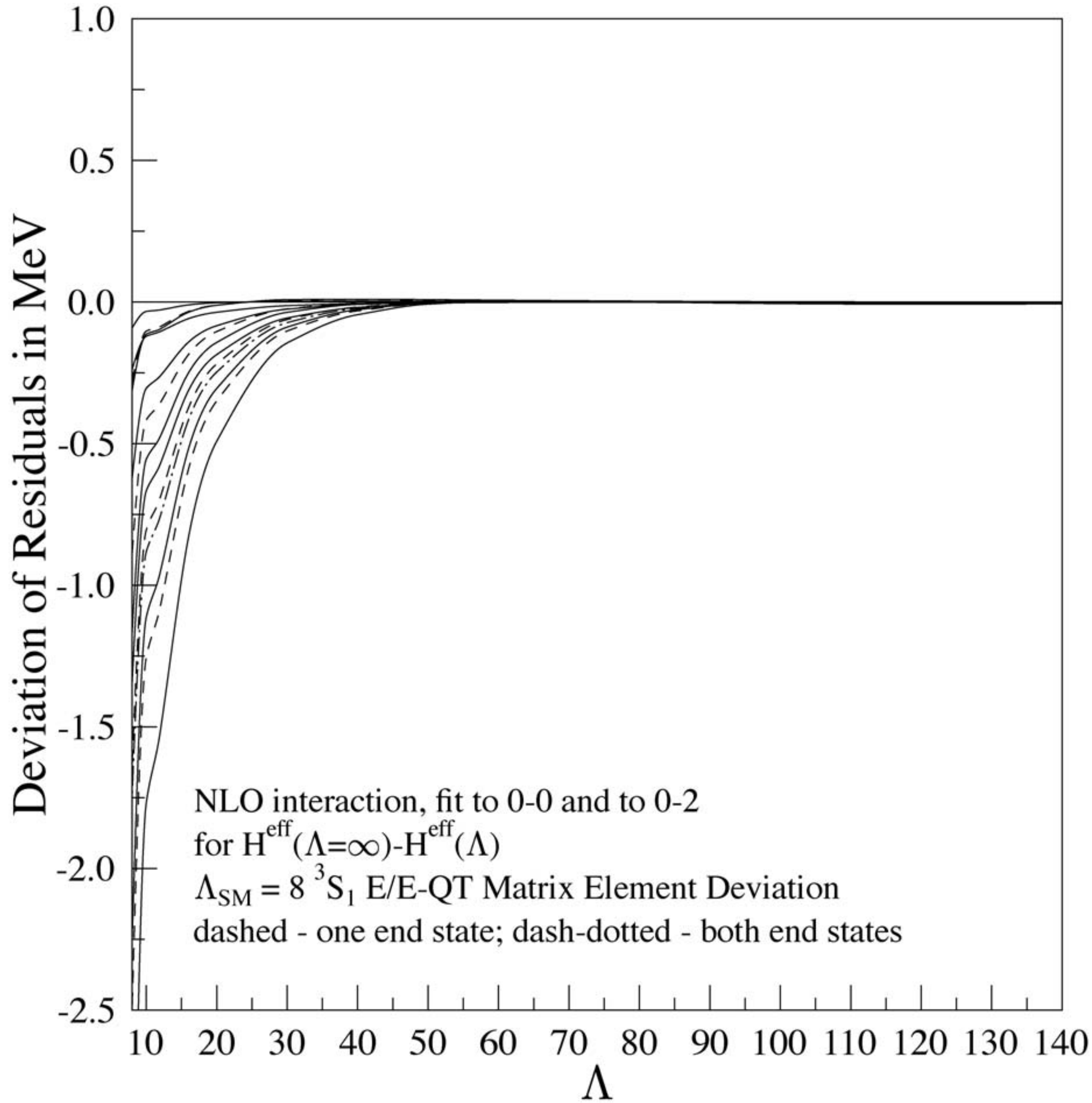

Deviation of Residuals in MeV
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
NLO interaction, fit to 0-0 and to 0-2
for H^eff(Λ=∞)-H^eff(Λ)
Λ_SM = 8 ^3S_1 E/E-QT Matrix Element Deviation
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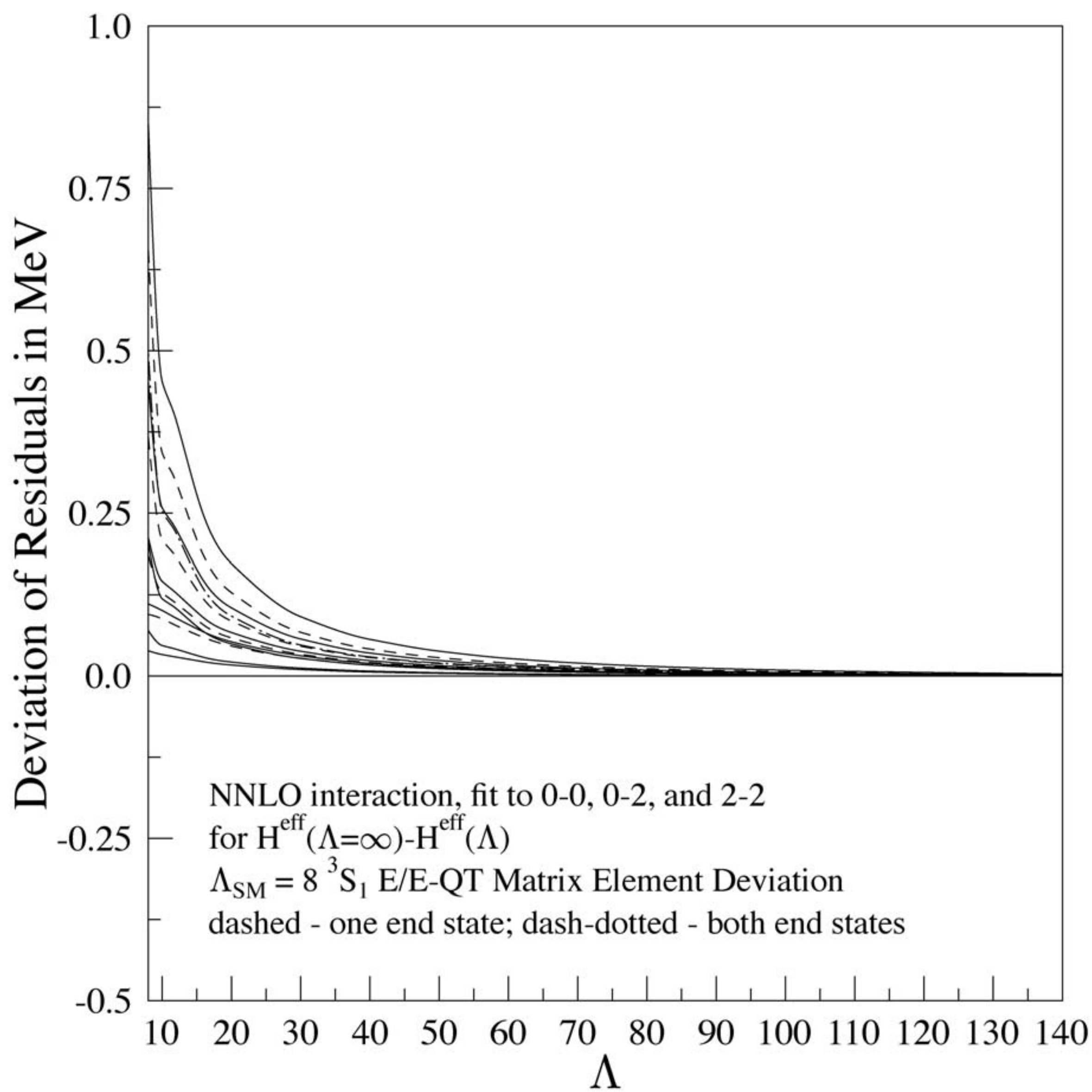

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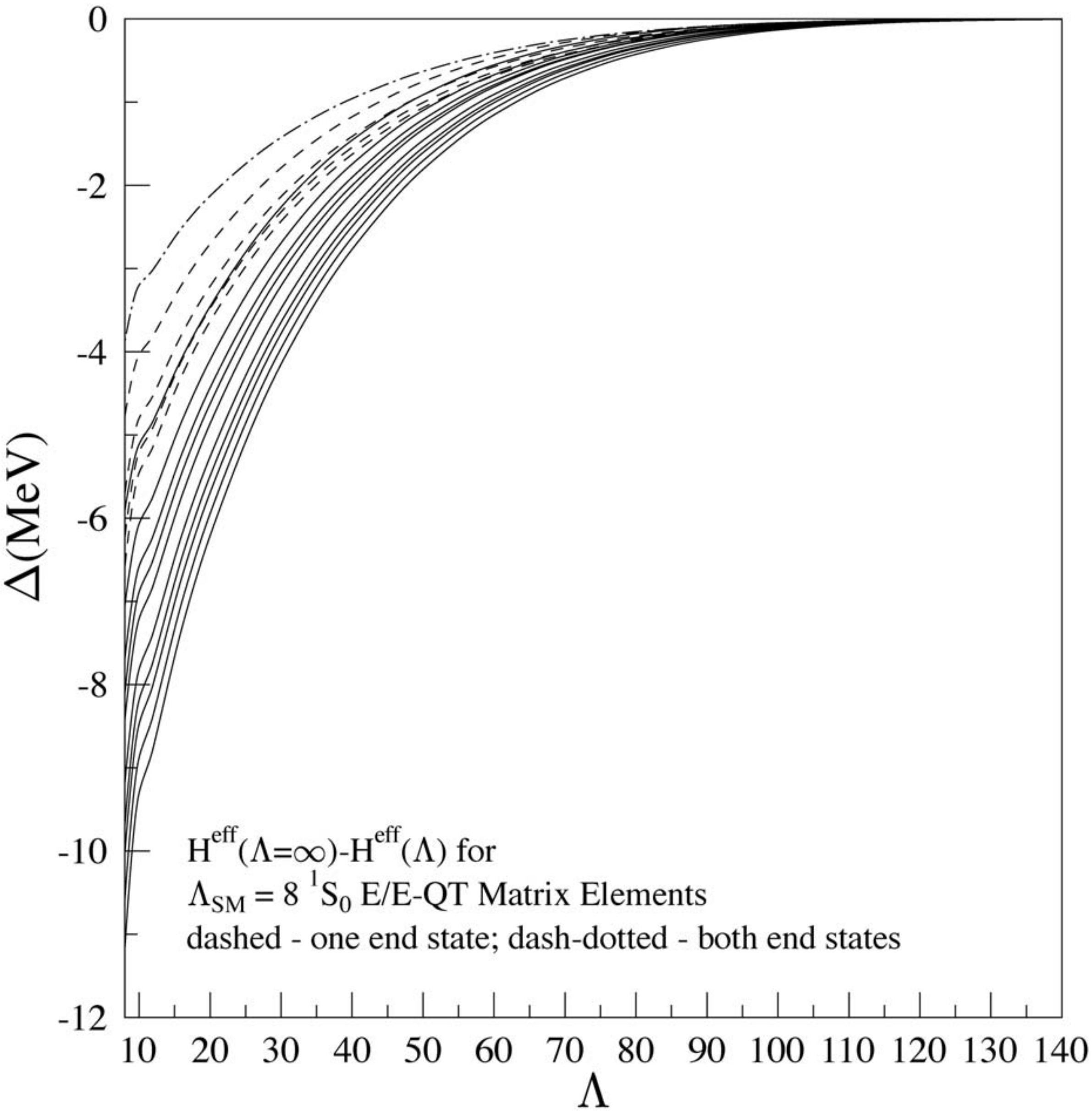

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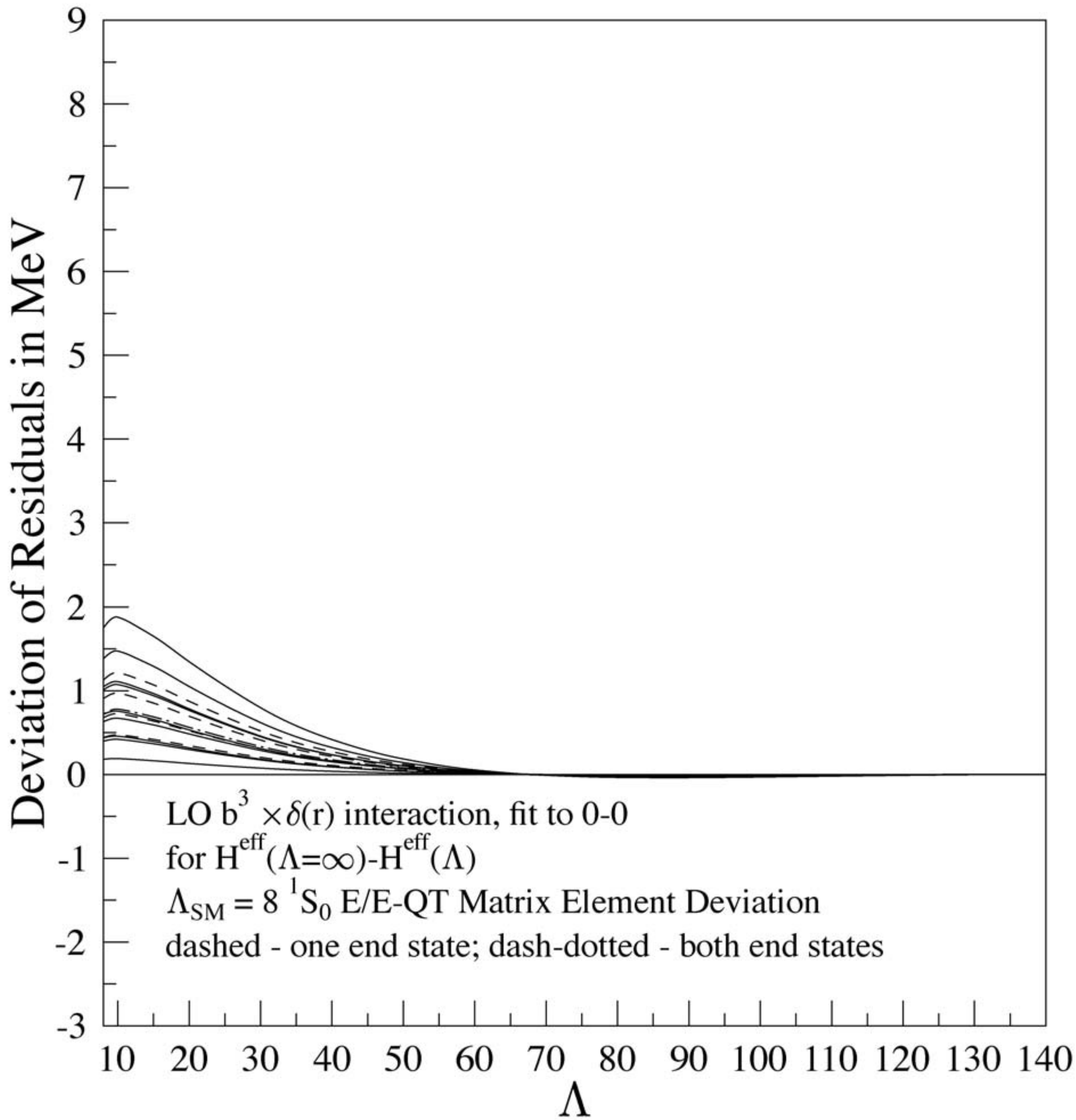
$\Lambda_{SM} = 8$ $^1S_0$ E/E-QT Matrix Elements
dashed - one end state; dash-dotted - both end states



Deviation of Residuals in MeV
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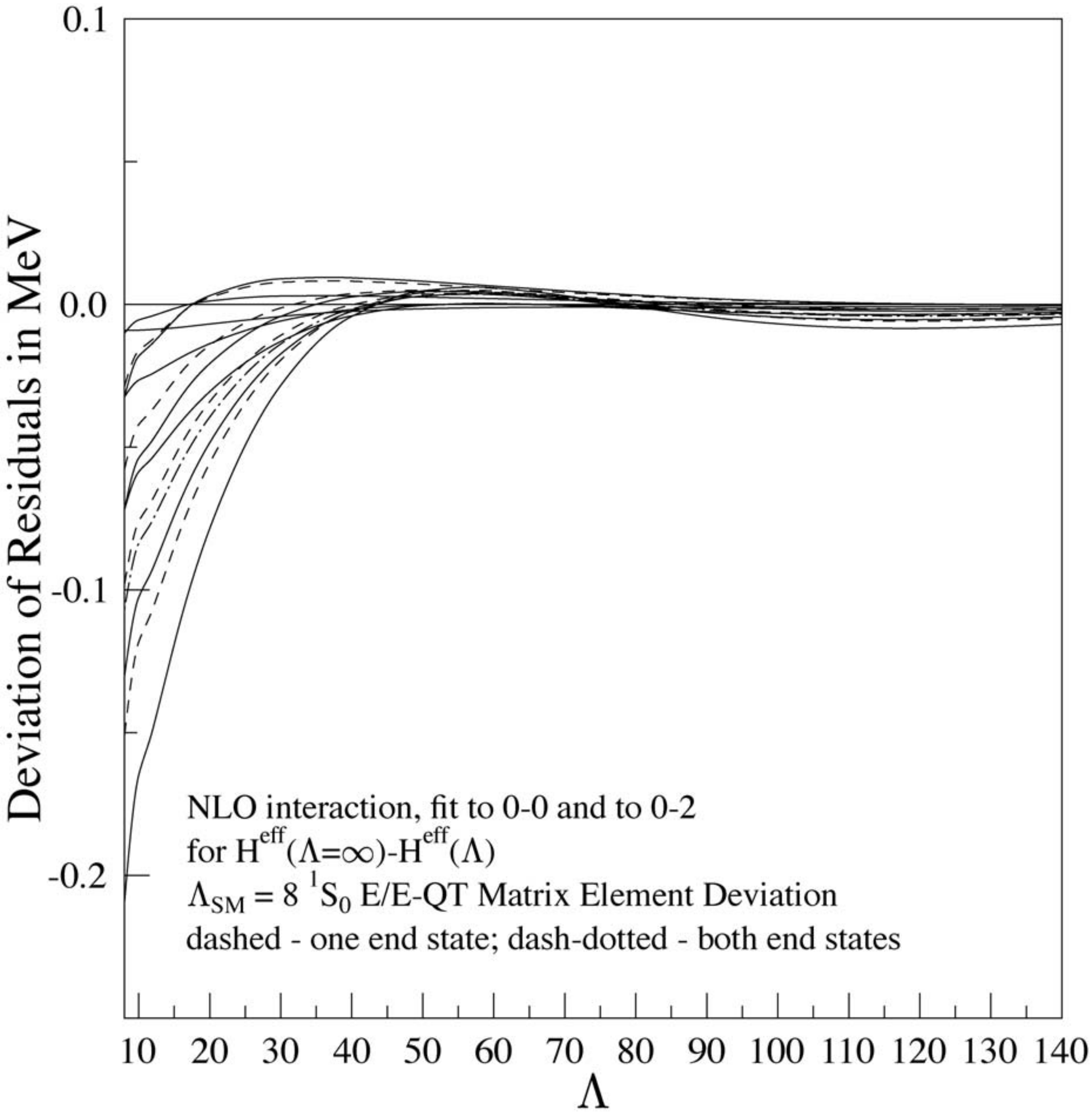
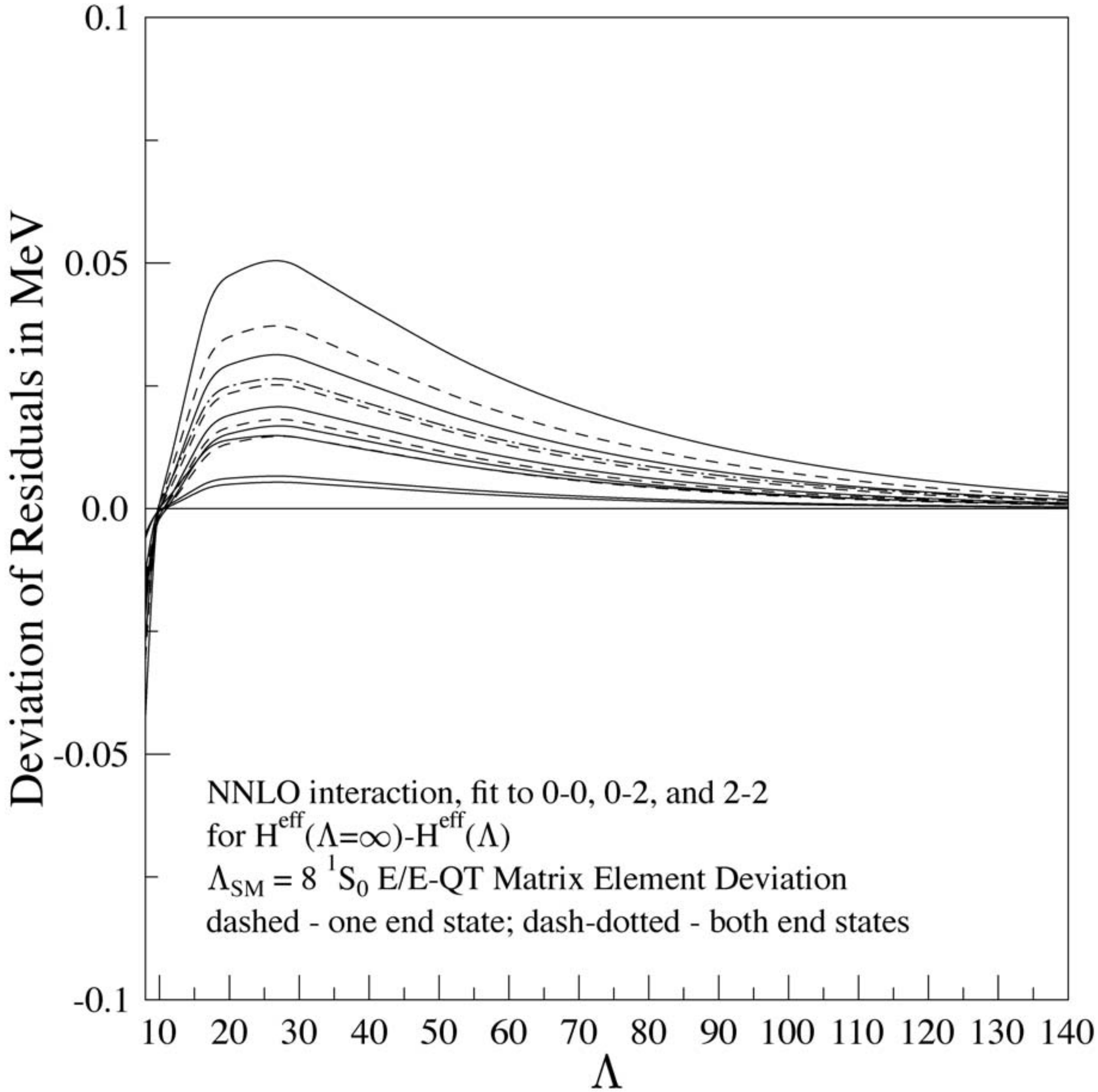
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
LO b^3 × δ(r) interaction, fit to 0-0
for H^eff(Λ=∞)-H^eff(Λ)
Λ_SM = 8 ^1S_0 E/E-QT Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



0.1
0.0
-0.1
-0.2
Deviation of Residuals in MeV
10 20 30 40 50 60 70 80 90 100 110 120 130 140
Λ
NLO interaction, fit to 0-0 and to 0-2
for $H^{eff}(\Lambda=\infty)-H^{eff}(\Lambda)$
$\Lambda_{SM} = 8$ $^1S_0$ E/E-QT Matrix Element Deviation
dashed - one end state; dash-dotted - both end states



Deviation of Residuals in MeV
0.1
0.05
0.0
-0.05
-0.1
10
20
30
40
50
60
70
80
90
100
110
120
130
140
Λ
NNLO interaction, fit to 0-0, 0-2, and 2-2
for H^eff(Λ=∞)-H^eff(Λ)
Λ_SM = 8 ¹S₀ E/E-QT Matrix Element Deviation
dashed - one end state; dash-dotted - both end states

Summary: Formulating HOBET and relating it to the SM

- The HO SM's energy-based Q leads to high-momentum $P - Q$ coupling responsible for nonperturbative behavior in two-body G
- These effects can be removed by a resummation of QT
- The same effects confuse an association of the short-range operator VGV with the standard plane-wave contact-gradient expansion
- This can be addressed by a redefining of the contact-gradient expansion to remove operator mixing
  - ◇ VGV then can be isolated in the deep SM states
  - ◇ for a system with enough bound states, the coefficients of the contract-gradient expansion could be fully determined - removing the hard core

- From the SM perspective (true HO states) this instructs one to renormalize the contact-gradient expansion in a defined way for edge states
  - ◇ generic result, e.g., required for $V - low - k$
- Physics governed by $G_0 = \frac{1}{E - \frac{1}{2M}(\vec{k}_1^2 + ... + \vec{k}_{A-1}^2)}$
  - ◇ very physical: extended Jacobi coordinate for "halo" states
  - ◇ effectively isolates all E-dependence in BH
  - ◇ has implications for Lee-Suzuki done at cluster level: the extended Jacobi coordinate is generally not present

- Discussion done from SM viewpoint; from ET viewpoint, corresponds to the choice of a new P-space, soft and CM-invariant

$$P_0 = \sum_{P_0} |n\rangle\langle n| \rightarrow P'(E) = \sum_{P'_0} |\tilde{n}\rangle\langle\tilde{n}|$$

normalized so the the $\{|\tilde{n}\rangle\ \}$ basis remains orthonormal

$$|\tilde{n}\rangle = \frac{\frac{1}{E-Q_0T}|n\rangle}{\sqrt{\langle n|\frac{1}{E-TQ_0}\frac{1}{E-Q_0T}|n\rangle}}$$

- P' is asymptotically correct
- A well-behaved $H_{eff} = H + H\frac{1}{E-Q'H}Q'H$

- Intriguing question: analytically continuing into continuum
  - ◇ Would allow one to go directly from scattering data to the HOBET appropriate for a given Λ_{SM}, b
  - ◇ e.g., for deuteron, all we can do now, independent of $av18$, is to determine a_{LO}
  - ◇ with $av18$, our computed VGV matrix elements “encoded” NN phase shifts in the m.e.’s we studied
  - ◇ can we avoid all of this work?