Reactions Theory I

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Classification by Outcome

1. **Elastic scattering:**
   projectile and target stay in their g.s.
2. **Inelastic scattering:**
   projectile or target left in excited state
3. **Transfer reaction:**
   1 or more nucleons moved to the other nucleus
4. **Fragmentation/Breakup/Knockout:**
   3 or more nuclei/nucleons in the final state
5. **Charge Exchange:**
   A is constant but Z (charge) varies, e.g. by pion exchange
6. **Multistep Processes:**
   intermediate steps can be any of the above
   ('virtual' rather than 'real')

7. **Deep inelastic collisions:**
   Highly excited states produced
8. **Fusion:**
   Nuclei stick together
9. **Fusion-evaporation:**
   fusion followed by particle-evaporation and/or gamma emission
10. **Fusion-fission:**
    fusion followed by fission

The first 6 processes are *Direct Reactions* (DI)
The last 3 processes give a *Compound Nucleus* (CN).

Compound and Direct Reactions

So when two nuclei collide there are 2 types of reactions:

1. Nuclei can coalesce to form highly excited **Compound nucleus (CN)** that lives for relatively long time.
   Long lifetime sufficient for excitation energy to be shared by all nucleons. If sufficient energy localised on one or more nucleons (usually neutrons) they can escape and CN decays.
   Independence hypothesis: CN lives long enough that it loses its memory of how it was formed. So probability of various decay modes independent of entrance channel.
2. Nuclei make 'glancing' contact and separate immediately, said to undergo **Direct reactions(DI)**.
   Projectile may lose some energy, or have one or more nucleons transferred to or from it.

Location of reactions:

CN reactions at small impact parameter,

DI reactions at surface & large impact parameter.

CN reaction involves the whole nucleus.

DI reaction usually occurs on the surface of the nucleus. This leads to diffraction effects.

Usually both DI and CN may contribute to the same reaction.

Duration of reactions:

A typical nucleon orbits within a nucleus with a period of $\sim\ 10^{-22}$ sec. [as K.E. $\sim$ 20 MeV].

If reaction complete within this time scale or less then no time for distribution of projectile energy around target $\Rightarrow$ DI reaction occurred. CN reactions require $\gg\ 10^{-22}$ sec.

Angular distributions:

In DI reactions differential cross section strongly forward peaked as projectile continues to move in general forward direction.

Differential cross sections for CN reactions do not vary much with angle (not complete isotropy as still slight dependence on direction of incident beam).

Types of direct reactions:

Can identify various types of DI processes that can occur in reactions of interest:

1. **Elastic scattering**: $A(a,a)A$ with internal states unchanged. Zero Q-value.

2. **Inelastic scattering**: $A(a,a')A^*$ or $A(a,a^*)A^*$.
   Projectile a gives up some of its energy to excite target nucleus A. If nucleus a also complex nucleus, it can also be excited.

   [If energy resolution in detection of a not small enough to resolve g.s. of target from low-lying excited states then cross section will be sum of elastic and inelastic components. This is called **quasi-elastic scattering**].

3. **Breakup reactions**: Usually referring to breakup of projectile *a* into two or more fragments. This may be **elastic** breakup or **inelastic** breakup depending on whether target remains in ground state.

4. **Transfer reactions**:
   Stripping: transfer from projectile.
   Pickup: transfer to projectile.

5. **Charge exchange reactions**: mass numbers remain the same. Can be elastic or inelastic.

Some terminology

Reaction channels:

In nuclear reaction, each possible combination of nuclei is called a **partition**.

Each partition further distinguished by state of excitation of each nucleus and each such pair of states is known as a **reaction channel**.

The initial partition, $a + A$ (both in their ground states) is known as the incident, or entrance channel. The various possible outcomes are the possible exit channels.

In a particular reaction, if not enough energy for a particular exit channel then it is said to be closed.

Model Calculations

Now for some Theory!

In quantum mechanics, models start with a **Hamiltonian** H:

$$\begin{aligned} \hat{H} &= T + V \qquad (1) \\ &= \text{kinetic energy} + \text{potential energy} \end{aligned}$$

Then solve a Schrödinger equation (SE) for total energy E:

$$\hat{H}\Psi = E\Psi \qquad (2)$$

Boundary conditions are made for Ψ according to the reaction.
Predictions are based on probabilities $|\Psi|^2$.

Elastic Scattering from Spherical Potentials

Non-relativistic, 2-body formalism of Schrödinger equation (SE).
Look at 2-body system in potential $V(r)$

$$\begin{aligned} \mathbf{r} &= (\mathbf{r}_1 - \mathbf{r}_2) \\ \mathbf{R} &= (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \end{aligned}$$

The time-indendent Schrödinger equation is

$$\hat{H}\Psi = E\Psi \tag{3}$$

The Hamiltonian for the system is

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2m_1}\nabla_{\mathbf{r1}} - \frac{\hbar^2}{2m_2}\nabla_{\mathbf{r2}} + V(r) \\ &= -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}} - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}} + V(r) \end{aligned} \tag{4}$$

[$m = m_1 m_2/(m_1 + m_2)$ and $M = m_1 + m_2$]

Phase Shifts from Potentials

Thus can look for separable solutions of the form

$$\Psi(\mathbf{R},\mathbf{r}) = \phi(\mathbf{R})\psi(\mathbf{r}) \tag{5}$$

Substituting for Ψ back in SE (3) gives LHS function of $\mathbf{R}$ and RHS function of $\mathbf{r}$. Thus both equal to common constant, E_{cm}. Hence

$$- \frac{\hbar^2}{2M}\nabla^2_{\mathbf{R}}\ \phi(\mathbf{R}) \ = \ E_{cm}\ \phi(\mathbf{R}) \tag{6}$$

and

$$\left(- \frac{\hbar^2}{2m}\nabla^2_{\mathbf{r}} + V(r) \right) \psi(\mathbf{r}) \ = E_{rel}\ \psi(\mathbf{r}) \tag{7}$$

where $\quad E_{rel} \ = \ E \ - \ E_{cm}.$

In scattering, if m_1 is projectile incident on stationary target m_2 then

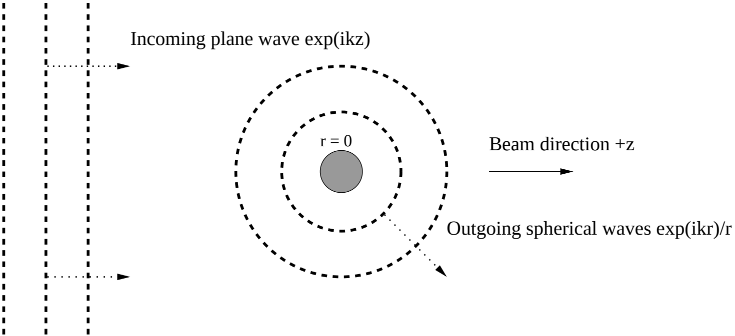
$$E_{cm} = \frac{m_1}{m_1 + m_2} E$$

$$E_{rel} = \frac{m_2}{m_1 + m_2} E$$

Solution to (6) is simple: $\phi(\mathbf{R}) = A\, e^{i\mathbf{K}\cdot\mathbf{R}}$ which is plane wave. Thus c.o.m. moves with constant momentum $\hbar\mathbf{K}$ and does not change after scattering. (Note, $E_{cm} = \hbar^2 K/2M$).

The real physics is in Eq.(7).

Spherical Potentials: $\psi(\mathbf{r})$ from $V(r)$



If incident beam $\sim$ 1 cm wide, this is 10^{13} fm $= 10^{12} \times$ nuclear size).

Thus beam can be represented by **plane wave** $e^{i\mathbf{k}\cdot\mathbf{r}}$

Phase Shifts from Potentials

As $|\mathbf{r}| \to \infty$ (i.e. moving away radially from scattering centre),

$$\psi(\mathbf{r}) \ \to \ N \left(e^{i\mathbf{k}\cdot\mathbf{r}} \ + \ f(\theta,\varphi) \ \frac{e^{ikr}}{r} \right) \tag{8}$$

where k is defined as $E_{rel} = \hbar^2 k^2/2m$. Take $N = 1$.

In QM, flux (probability current density) is given by

$$\mathbf{J} = Re \left[\psi^* \left(-\frac{i\hbar}{m} \nabla_{\mathbf{r}} \right) \psi \right]$$

For incident flux, $\psi_{inc} = e^{i\mathbf{k}\cdot\mathbf{r}}$ and

$$\begin{aligned} \mathbf{J}_{inc} &= Re \left[e^{-i\mathbf{k}\cdot\mathbf{r}} \left(-\frac{i\hbar}{m} \nabla_{\mathbf{r}} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \right] \\ &= \frac{\hbar \mathbf{k}}{m} \ . \end{aligned} \tag{9}$$

Cross Section

For scattered flux $\psi_{scat} = f(\theta, \varphi)\ \frac{e^{ikr}}{r}$ and hence we obtain

$$\mathbf{J}_{scat} = \mathrm{J}_{inc}\ \frac{|f(\theta,\varphi)|^2}{r^2}\ \hat{\mathbf{r}} \tag{10}$$

Define the **differential cross-section** (in units of area) as

The number of particles scattered into unit solid angle per unit time, per unit incident flux, per target point,

so
$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2. \tag{11}$$

What does $f(\theta, \varphi)$ look like?

We know what the solution must look like asymptotically (outside potential):

$$\psi(\mathbf{r}) \;\to\; N\left(e^{i\mathbf{k}\cdot\mathbf{r}} \;+\; f(\theta,\varphi)\,\frac{e^{ikr}}{r}\right) \tag{12}$$

For $V(r)$ a central potential, expand partial wave solutions as

$$\psi(\mathbf{r}) = \sum_{\ell} \frac{u_\ell(r)}{kr}\, Y_{\ell 0}(\theta) \tag{13}$$

for *all* radii. Choose z-axis along incident beam, so $e^{i\mathbf{k}\cdot\mathbf{r}} \;=\; e^{ikz}$.

The partial-wave radial Schrödinger equation is

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{2m}{\hbar^2} V(r) - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = 0 \tag{14}$$

Asymptotic Solutions

Choose $V(r) = 0$ for $r > r_0$. Beyond r_0 get **free solution**

$$u_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = 0 \qquad (15)$$

Free solution is related to Coulomb functions

$$\begin{array}{llllll} r > r_0: & u_\ell = A_\ell & F_\ell(kr) & + B_\ell & G_\ell(kr) & (16) \\ & & \uparrow & & \uparrow & \\ & & kr\, j_\ell(kr) & & -kr\, n_\ell(kr) & \\ & & \text{(regular)} & & \text{(irregular)} & \\ r \to \infty: & & \to \sin(kr - \ell\pi/2) & & \cos(kr - \ell\pi/2) & \end{array}$$

Phase Shifts

As $r \to \infty$

$$\begin{aligned} u_\ell &\to A_\ell \sin(kr - \ell\pi/2) + B_\ell \cos(kr - \ell\pi/2) \\ &= C_\ell \sin(kr - \ell\pi/2 + \delta_\ell) \end{aligned} \quad (17)$$

where δ_ℓ is known as the **phase shift**, so $\tan\delta_\ell = B_\ell/A_\ell$ here.

If $V = 0$ then solution must be valid everywhere, even at origin where it has to be regular. Thus $B_\ell = 0$.

So, asymptotically (long way from scattering centre):

$$\begin{aligned} \text{For } V = 0 \qquad & u_\ell = A_\ell \sin(kr - \ell\pi/2) \\ \text{and for } V \neq 0 \qquad & u_\ell = C_\ell \sin(kr - \ell\pi/2 + \delta_\ell) \end{aligned} \quad (18)$$

Thus, switching scattering potential ‘on’ shifts the phase of the wave function at large distances from the scattering centre.

Scattering amplitudes from Phase Shifts

Now substituting for u_ℓ from Eq.(18) back into Eq.(13) for $\psi(\mathbf{r})$, and after some angular momentum algebra, we obtain a scattering wave function which, when equated with the required asymptotic form of Eq.(12) gives

$$f(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)\ T_\ell\ P_\ell(\cos\theta) \tag{19}$$

$$\text{where} \quad T_\ell = e^{i\delta_\ell}\ \sin\delta_\ell = \frac{1}{2i}(S_\ell - 1)\ . \tag{20}$$

T_ℓ is the partial wave T-matrix.
S_ℓ is the partial-wave S-matrix.
They are connected to the **T-matrix** (see later).

There is no dependence on φ because of *central* potentials.

Properties of the S-matrix S_ℓ:

The S-matrix element is a complex number $S_\ell = e^{2i\delta_\ell}$

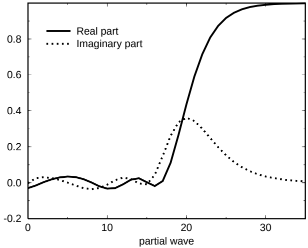
1. For purely diffractive (real) potentials $|S_\ell| = 1$.
   This is called *unitarity*, and is the conservation of flux.
   δ_ℓ is usually positive for attractive potentials.
2. For absorptive (complex) potentials, $|S_\ell| \leq 1$.
   The total absorption=fusion cross section is

$$\sigma_A = \frac{\pi}{k} \sum_\ell (2\ell + 1)(1 - |S_\ell|^2) \tag{21}$$

ℓ-dependence of the S-matrix S_ℓ:

1. The ℓ value (partial wave) where $Re(S_\ell) \sim 0.5$ is the **grazing** ℓ value.
2. Partial wave ℓ related to impact parameter b in semiclassical limit: $\ell = k\ b$

Typical dependence of S_ℓ on ℓ for scattering from an absorptive optical potential:

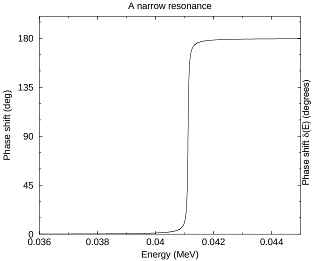


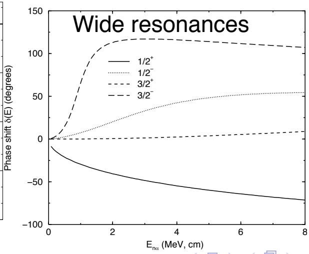
Absorption when $|S_\ell|^2 < 1$ in the interior.

Resonances:

Occur when two particles trapped together (eg for time τ).

1. Give energy peaks in cross sections with fwhm $\Gamma \sim \hbar/\tau$.
2. Phase shifts typically rise rapidly through $\pi/2$ (90°) as $\delta_{\rm res}(E) = \arctan\left(\frac{\Gamma/2}{E_r - E}\right) + \delta_{bg}$ for peak at energy E_r.
3. Cross section peak $\sigma(E) = \frac{4\pi}{k^2}(2L+1)\frac{\Gamma^2/4}{(E-E_r)^2+\Gamma^2/4} + \sigma_{bg}$





Free Green's function $G_0(E)$

Can write the Schrödinger equation as

$$(E - H)\,\psi \;=\; 0 \qquad \text{or} \qquad (E - H_0)\,\psi \;=\; V\,\psi \tag{22}$$

where $H = H_0 + V$. Thus

$$\psi \;=\; (E - H_0)^{-1}\,V\,\psi \;=\; G_0(E)\,V\,\psi \tag{23}$$

$G_0(E)$ is the **Green's operator**.

Eq.(23) is not general solution for ψ as can add on solution of homogeneous equation: the plane wave χ present when $V = 0$.

$$(E \;-\; H_0)\,\chi \;=\; 0 \tag{24}$$

Lipmann-Schwinger equation

General solution of Eq.(22) is therefore

$$\psi = \chi + G_0(E)\, V\, \psi \tag{25}$$

This is iterative

$$\psi = \chi + G_0\, V\, \chi + G_0\, V\, G_0\, V\, \chi + \ldots. \tag{26}$$

Eq.(25) can be written in integral form as the **Lipmann-Schwinger** equation

$$\psi(\mathbf{r}) = \chi(\mathbf{r}) + \int d\mathbf{r}'\, G(\mathbf{r},\mathbf{r}')\, V(r')\, \psi(\mathbf{r}') \tag{27}$$

where $G(\mathbf{r},\mathbf{r}')$ is the **Green's function**.

Integral expression for the Scattering Amplitude

The $\chi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ = incident plane wave,

and we use $\psi^{(+)}_{\mathbf{k}}(\mathbf{r})$ for the scattering wave function.

(i.e. incident momentum $\mathbf{k}$ and $(+)$ for outgoing waves solution).

Comparing Eq.(27) with required asymptotic form for ψ we see that integral term must tend to

$$f(\theta)\,\frac{e^{ikr}}{r} \qquad \text{as} \quad |\mathbf{r}| \to \infty \quad . \tag{28}$$

Thus, using properties of Green's function, with $\mathbf{k}'$ at angle θ,

$$f(\theta) \;=\; -\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\, V(r)\,\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) \quad . \tag{29}$$

Transition matrix element

In Dirac (bra-ket) notation we write this

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{k}' \mid V \mid \psi^{(+)}_{\mathbf{k}} \rangle \tag{30}$$

$$= -\frac{m}{2\pi\hbar^2} T(\mathbf{k}', \mathbf{k}) \quad . \tag{31}$$

$T(\mathbf{k}', \mathbf{k})$ is known as the **Transition matrix element**.
So the angular cross section is

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} |T(\mathbf{k}', \mathbf{k})|^2. \tag{32}$$