Nuclear Reaction Theory: concepts and applications – Part II

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Reactions: rarest beams - ‘few’ nuclei per second

- Fast exotic beams allow for
  - thick secondary targets
  - event-by-event identification
  - clean product selection
  - nevertheless …..

- Example
  - $s = 100$ millibarn
  - $N_T = 10^{21}$
  - $N_B = 3$ Hz
  - $N_R = 26$/day
    - $= 3 \times 10^{-4}$ Hz

\[ N_R = s \times N_T \times N_B \]

- $s$ cross section
- $N_T$ atoms in target
- $N_B$ beam rate
- $N_R$ reaction rate

From Thomas Glasmacher
Point particles: partial wave S-matrix

Scattering states

\[ E_{cm} > 0 \quad k = \sqrt{\frac{2\mu E_{cm}}{\hbar^2}} \]

\[ \left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} - \frac{2\mu}{\hbar^2} U_{\ell j}(r) + k^2 \right) u_{k\ell j}(r) = 0 \]

and beyond the range of the nuclear forces, then

\[ \left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} - \frac{2\eta k}{r} + k^2 \right) u_{k\ell j}(r) = 0, \quad \eta = \frac{\mu Z_c Z_v e^2}{\hbar k} \]

\[ F_{\ell}(\eta, kr), \quad G_{\ell}(\eta, kr) \] regular and irregular Coulomb functions

\[ u_{k\ell j}(r) \rightarrow e^{i\delta_{\ell j}} [\cos \delta_{\ell j} F_{\ell}(\eta, kr) + \sin \delta_{\ell j} G_{\ell}(\eta, kr)] \]

\[ \rightarrow (i/2) [H_{\ell}^{(-)}(\eta, kr) - S_{\ell j} H_{\ell}^{(+)}(\eta, kr)] \]

\[ H_{\ell}^{(\pm)}(\eta, kr) = G_{\ell}(\eta, kr) \pm iF_{\ell}(\eta, kr) \]
Phase shift and partial wave S-matrix: Recall

\[ u_{k\ell j}(r) \rightarrow e^{i\delta_{\ell j}} [\cos \delta_{\ell j} F_\ell(\eta, kr) + \sin \delta_{\ell j} G_\ell(\eta, kr)] \]

If \( U(r) \) is real, the phase shifts \( \delta_{\ell j} \) are real, and […] also

\[ u_{k\ell j}(r) \rightarrow (i/2) \left[ H_\ell(\eta, kr) - S_{\ell j} H_\ell(\eta, kr) \right] \]

\[ S_{\ell j} = e^{2i\delta_{\ell j}} \]

\[ |S_{\ell j}|^2 \quad \text{survival probability in the scattering} \]

\[ (1 - |S_{\ell j}|^2) \quad \text{absorption probability in the scattering} \]

Having calculate the phase shifts and the partial wave S-matrix elements we can then compute all scattering observables for this energy and potential (but later).
Ingoing and outgoing waves amplitudes

\[ u_{k\ell}(r) \rightarrow (i/2) \left[ 1 H^{(-)}_\ell - S_\ell H^{(+)}_\ell \right] \]
Eikonal approximation: for point particles (1)

Approximate (semi-classical) scattering solution of

\[
\left(-\frac{\hbar^2}{2\mu} \nabla^2_r + U(r) - E_{cm}\right) \chi^+_k(\vec{r}^*) = 0, \quad \mu = \frac{m_cm_v}{m_c + m_v}
\]

\[
\left(\nabla^2_r - \frac{2\mu}{\hbar^2} U(r) + k^2\right) \chi^+_k(\vec{r}^*) = 0
\]

valid when \(|U|/E \ll 1, \quad ka \gg 1\) \rightarrow high energy

Key steps are: (1) the distorted wave function is written

\[
\chi^+_k(\vec{r}^*) = \exp(i\vec{k} \cdot \vec{r}^*) \omega(\vec{r}^*)
\]

all effects due to \(U(r)\), modulation function

(2) Substituting this product form in the Schrödinger Eq.

\[
\left[2i\vec{k} \cdot \nabla \omega(\vec{r}^*) - \frac{2\mu}{\hbar^2} U(r)\omega(\vec{r}^*) + \nabla^2\omega(\vec{r}^*)\right] \exp(i\vec{k} \cdot \vec{r}^*) = 0
\]
Eikonal approximation: point particles (2)

\[
\left[ 2i\vec{k} \cdot \nabla \omega(\vec{r}) - \frac{2\mu}{\hbar^2} U(r)\omega(\vec{r}) + \nabla^2 \omega(\vec{r}) \right] \exp(i\vec{k} \cdot \vec{r}) = 0
\]

The conditions \( |U|/E \ll 1, \quad ka \gg 1 \) imply that

\[
2\vec{k} \cdot \nabla \omega(\vec{r}) \gg \nabla^2 \omega(\vec{r})
\]

Slow spatial variation cf. \( k \)

and choosing the z-axis in the beam direction \( \vec{k} \)

\[
\frac{d\omega}{dz} \approx -\frac{i\mu}{\hbar^2 k} U(r)\omega(\vec{r})
\]

with solution

\[
\omega(\vec{r}) = \exp \left[ -\frac{i\mu}{\hbar^2 k} \int_{-\infty}^{z} U(r)dz' \right]
\]

1D integral over a straight line path through U at the impact parameter b
Eikonal approximation: point particles (3)

\[ \chi^{+}_{k} (\vec{r}) = \exp(i \vec{k} \cdot \vec{r}) \omega(\vec{r}) \approx \exp(i \vec{k} \cdot \vec{r}) \exp \left[ -\frac{i \mu}{\hbar^2 k} \int_{-\infty}^{\infty} U(r) dz' \right] \]

So, after the interaction and as \( z \to \infty \)

\[ \chi^{+}_{k} (\vec{r}) \to \exp(i \vec{k} \cdot \vec{r}) \exp \left[ -\frac{i \mu}{\hbar^2 k} \int_{-\infty}^{\infty} U(r) dz' \right] = S(b) \exp(i \vec{k} \cdot \vec{r}) \]

Eikonal approximation to the S-matrix \( S(b) \)

\[ S(b) = \exp \left[ -\frac{i}{\hbar v} \int_{-\infty}^{\infty} U(r) dz' \right] \]

Moreover, the structure of the theory generalises simply to few-body projectiles

\[ v = \frac{\hbar k}{m} \]

\( b \), \( r \), \( z \)
Eikonal approximation: point particles - summary

$$\chi^+_k(\vec{r}) = \exp(i \vec{k} \cdot \vec{r}) \exp \left[ -\frac{i \mu}{\hbar^2 k} \int_{-\infty}^{\infty} U(r) dz' \right]$$

$$\nu = \hbar k / m$$

$$\chi(b) = -\frac{1}{\hbar \nu} \int_{-\infty}^{\infty} U(r) dz$$

Limit of range of finite ranged potential

$$\chi^+_k(\vec{r}) \rightarrow S(b) \exp(i \vec{k} \cdot \vec{r})$$

$$S(b) = \exp[i \chi(b)] = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{\infty} U(r) dz' \right]$$
Semi-classical model for the S-matrix - $S(b)$

For high energy/or large mass, semi-classical ideas are good

$$kb \cong \ell, \text{ actually } \Rightarrow \ell + 1/2$$

$u_{k\ell j}(r) \rightarrow (i/2)[H_{\ell}^{(-)} - S_{\ell j}H_{\ell}^{(+)})$  

$S(b) = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{\infty} U(r)dz' \right]$
Point particle scattering – cross sections

All cross sections, etc. can be computed from the S-matrix, in either the partial wave or the eikonal (impact parameter) representation, for example (spinless case):

\[
\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) |1 - S_\ell|^2 \approx \int d^2\vec{b} \ |1 - S(b)|^2
\]

\[
\sigma_R = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) (1 - |S_\ell|^2) \approx \int d^2\vec{b} \ (1 - |S(b)|^2)
\]

\[
\sigma_{tot} = \sigma_{el} + \sigma_R = 2 \int d^2\vec{b} \ [1 - \text{Re}.S(b)]
\]

and where (cylindrical coordinates)

\[
\int d^2\vec{b} \equiv \int_0^\infty b db \int_0^{2\pi} d\phi = 2\pi \int_0^\infty b db
\]
What is involved in realistic reaction calculations?

\[ j_< = \ell - 1/2 \]
\[ j_> = \ell + 1/2 \]

\[ 23\text{O} \]
\[ Z=8 \]
\[ N=15 \]

\[ V_{\ell s}(r) \vec{\ell} \cdot \vec{s} \]

\[ V(r) + V_{so}(r) \vec{\ell} \cdot \vec{s} \]

\[ V_{so}(r) < 0 \]
Examples: What is involved – take neutron from $^{23}\text{O}$

Experimental separation energy known

Hartree-Fock mean field calculation
Neutron: proton: nucleon radial densities (HF)
Orientation I – neutron transfer – (p,d) reaction

Single neutron removal from $^{23}\text{O} = [1d_{5/2}]^6 [2s_{1/2}]$

- $2s_{1/2}$ $\Delta E = 2.73$ MeV
- $1d_{5/2}$ $\Delta E = 6.0$ MeV

$^{23}\text{O}(p,d)^{22}\text{O}(J^\pi_f)$

- $J^\pi_f = 0^+$, g.s.
- $S_n = 2.73$ MeV

- $J^\pi_f = 2^+$, 3.2 MeV
- $S_n = 6.0$ MeV

transfer reaction code(s) available at:
http://www.nucleartheory.net/NPG/code.htm
Transfer reaction transition amplitudes - DWBA

\[ T(p, d) = \langle \chi_{d, \vec{k}_d}^{(-)} \Phi(A, J_f) \phi_d | V_{np} | \chi_{p, \vec{k}_p}^{(+)} \Phi(A + 1, J_i) \rangle \]

\[ V_{np} \phi_d(\vec{r}) \approx D_0 \delta(\vec{r}) \text{ - short range} \]

\[ \phi_{n\ell j}(\vec{r}_n) = \langle \Phi(A, J_f) | \Phi(A + 1, J_i) \rangle \]
Neutron bound state wave functions

$r_R = r_C = r_{so} \approx 1.25\text{fm}$

$a_R = a_{so} \approx 0.7\text{fm}$  

$V_{so} = 6\text{MeV}$

radius (fm)

radial wave function (fm$^{-1/2}$)

1$d_{5/2}$

2$s_{1/2}$

$^{23}\text{O}$
Global optical potentials – e.g. CH91 for nucleons

A GLOBAL NUCLEON OPTICAL MODEL POTENTIAL*

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Theoretical nucleon potential – based on density

\[ V_{NB}(R) = \int dr_2 \rho_B(r_2) v_{NN}(R + r_2) \]

include the effect of NN interaction in the “nuclear medium” – Pauli blocking of pair scattering into occupied states (e.g. M3Y, JLM)

But as \( E \to \) high

\[ v_{NN} \to v_{NN}^{\text{free}} \]
For finite nuclei, what value of density should be used in calculation of nucleon-nucleus potential? Usually the local density at the mid-point of the two nucleon positions \( \mathbf{r}_x \).

\[
U_B(R) = V_B(R) + iW_B(R) = \int d\mathbf{r}_2 \rho_B(r_2) \frac{U(E, \rho(r_x))}{\rho(r_x)} f(r)
\]
JLM interaction – fine tuning

Strengths of the real and imaginary parts of the potential can be adjusted based on experience of fitting data.

\[ U_B(R) = \lambda_v V_B(R) + i \lambda_w W_B(R) \]

\[ \lambda_v \approx 1.0 \]
\[ \lambda_w \approx 0.8 \]

\( p + {}^{16}\text{O} \)

JLM folded nucleon-nucleus optical potentials

\[ \lambda_v \approx 1.0 \]
\[ \lambda_w \approx 0.8 \]

Transfer reaction transition amplitudes - DWBA

\[ T(p, d) = \langle \chi^{(-)}_{d, k_d} \Phi(A, J_f) \phi_d | V_{np} | \chi^{(+)}_{p, k_p} \Phi(A + 1, J_i) \rangle \]

exit channel

\[ V_{np} \phi_d(\vec{r}) \approx D_0 \delta(\vec{r}) \quad \text{- short range} \]

\[ \phi_{n\ell j}(\vec{r}_n) = \langle \Phi(A, J_f) | \Phi(A + 1, J_i) \rangle \]

entrance channel
Global optical potentials – e.g. for deuterons

Physical Review C, Volume 21, Number 6, June 1980

Global optical model potential for elastic deuteron scattering from 12 to 90 MeV

W. W. Daehnick, J. D. Childs,* and Z. Vrcełj

Map of Targets and Energies Searched

![Graph showing scattering data](image)

[Graph legend with various isotopes and energy levels]
Calculated \((p,d)\) transfer (pick-up) cross sections

\[ ^{23}\text{O} \]

\[ 1/2^+, \text{Chapel–Hill} \]
\[ 1/2^+, \text{JLM G–matrix} \]
\[ 5/2^+, \text{Chapel Hill} \]

Cross section (mb/sr)

C.m. angle (degrees)
Example – from Ian Thompson EBSS 2011 slides
Experiments do not measure target final states. Final state of core c measured – using decay gamma rays.

How to describe and what can we learn from these?

Another experimental option is one-nucleon removal – at ~100 MeV/nucleon and greater – fragmentation beams.

Orientation II – neutron removal – or knockout

12C

Experiments do not measure target final states. Final state of core c measured – using decay gamma rays.

J_i

J_f

Use for reactions – stripping/knockout of a nucleon

\[ \sigma_{\text{strip}} = \int \, \mathrm{d}b \, \langle \phi_0 | S_c(b_c) |^2 (1 - |S_1(b_1)|^2) | \phi_0 \rangle \]

Have also assumed the sudden/adiabatic approximation
Effective interactions – Folding models

Double folding

\[ U_{AB}(R) = \int dr_1 \int dr_2 \rho_A(r_1) \rho_B(r_2) v_{NN}(R + r_1 - r_2) \]

Single folding

\[ U_B(R) = \int dr_2 \rho_B(r_2) v_{NN}(R - r_2) \]
Core-target effective interactions – for $S_{c}(b_{c})$

Double folding

$U_{AB}(R) = \int dr_{1} \int dr_{2} \rho_{A}(r_{1}) \rho_{B}(r_{2}) t_{NN}(R + r_{2} - r_{1})$

$U_{AB}$

$\rho_{A}(r)$

$\rho_{B}(r)$

At higher energies – for nucleus-nucleus or nucleon-nucleus systems – first order term of multiple scattering expansion

$t_{NN}(r) = \left[ -\frac{\hbar v}{2} \sigma_{NN}(i + \alpha_{NN}) \right] f(r), \quad \int d\vec{r} f(r) = 1$

e.g. $f(r) = \delta(r)$

nucleon-nucleon cross section

resulting in a COMPLEX nucleus-nucleus potential

Diffractive (breakup) removal of a nucleon

\[ \sigma_{\text{diff}} = \int db \left\{ \langle \phi_0 | S_c S_v^2 | \phi_0 \rangle - |\langle \phi_0 | S_c S_v | \phi_0 \rangle|^2 \right\} \]
Orientation II – neutron removal – cross sections

Single neutron removal from $^{23}\text{O} \equiv [1d_{5/2}]^6 [2s_{1/2}]$

at 72 MeV/nucleon on a $^{12}\text{C}$ target

$[1d_{5/2}]^6 [2s_{1/2}]$

$2s_{1/2}$ 2.73 MeV

$1d_{5/2}$ 6.0 MeV

Gaussian density

rms radius 2.32 fm

$t_{NN}(r) = \left[ -\frac{\hbar v}{2} \sigma_{NN}(i + \alpha_{NN}) \right] f(r), \quad \int d\vec{r} f(r) = 1$
Eikonal S-matrix spatial selectivity

\[ |S_{22}(b_{22})|^2, (1-|S_n(b_n)|)^2 \]

impact parameters \( b_n, b_{22} \) (fm)

neutron

\(^{22}\text{O}\)

\(^{22}\text{O}\)
Neutron bound state wave functions

\[ r_R = r_C = r_{so} \approx 1.25\text{fm} \]

\[ a_R = a_{so} \approx 0.7\text{fm} \quad V_{so} = 6\text{MeV} \]
Orientation II – neutron knockout – cross sections

Single neutron removal from $^{23}$O $\equiv [1d_{5/2}]^6 [2s_{1/2}]$

- $2s_{1/2}$ $S_n=2.73$ MeV
- $1d_{5/2}$ $S_n=6.0$ MeV

$\sigma_{sp}(1d_{5/2})=28.3$ mb $\sigma_{sp}(2s_{1/2})=72.7$ mb

$\sigma_{-n} = 6 \sigma_{sp}(1d_{5/2}) + \sigma_{sp}(2s_{1/2}) = 242.5$ mb

Measurement at RIKEN [Kanungo et al. PRL 88 (‘02) 142502]
at 72 MeV/nucleon on a $^{12}$C target; $\sigma_{-n} = 233(37)$ mb
Sudden 1N removal from the mass A residue

Sudden removal: residue momenta probe the summed momenta of pair in the projectile rest rest frame

\[ \vec{K}_A = \frac{A}{A+1} \vec{K}_{A+1} - \vec{k}_1 \]

and component equations
Measurement of the residue’s momentum

Consider momentum components $p_{||}$ of the core parallel to the beam direction, in the projectile rest frame.

$$\Delta p \Delta z > \hbar/2$$
Forward momentum distributions of $^{22}\text{O}$ residues
End of Part II: thanks for your attention

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