Primer for Contemporary Quantum Field Theory in Hadron Physics at Nonzero Temperature and Density

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Preface

Hadron Physics lies at the interface between nuclear and particle (high energy) physics. Its focus is an elucidation of the role played by quarks and gluons in the structure of, and interactions between, hadrons. This was once particle physics but that has since moved to higher energy in search of a plausible grand unified theory and extensions of the so-called Standard Model. The only high-energy physicists still focusing on hadron physics are those performing the numerical experiments necessary in the application of lattice gauge theory, and those pushing at the boundaries of applicability of perturbative QCD or trying to find new kinematic regimes of validity.

There are two types of hadron: baryons and mesons: the proton and neutron are baryons; and the pion and kaon are mesons. Historically the names distinguished the particle classes by their mass but it is now known that there are structural differences: hadrons are bound states, and mesons and baryons are composed differently. Hadron physics is charged with the responsibility of providing a detailed understanding of the differences.

To appreciate the difficulties inherent in this task it is only necessary to remember that even the study of two-electron atoms is a computational challenge. This is in spite of the fact that one can employ the Schrödinger equation for this problem and, since it is not really necessary to quantise the electromagnetic field, the underlying theory has few complications.

The theory underlying hadron physics is quantum chromo-dynamics (QCD), and its properties are such that a simple understanding and simple calculations are possible only for a very small class of problems. Even on the domain for which a perturbative application of the theory is appropriate the final (observable) states in any experiment are always hadrons, and not

quarks nor gluons, so that complications arise in the comparison of theory with experiment.

The premier experimental facility for exploring the physics of hadrons is the Thomas Jefferson National Accelerator Facility (TJNAF), in Newport News, Virginia. Important experiments are also performed at the Fermi National Accelerator Facility (FermiLab), in Batavia, Illinois, near Argonne, and at the Deutsches Elektronensynchrotron (DESY) in Hamburg. These facilities use high-energy probes and/or large momentum-transfer processes to explore the transition from the nonperturbative to the perturbative domain in QCD. Other facilities also provide useful information; e.g., the Bates accelerator at MIT and MAMI in Mainz, but, typically, experiments at these lower-energy facilities do not probe inside hadrons, or an interpretation of their results in terms of quarks and gluons is hampered by our poor contemporary understanding of hadron bound state structure.

Modern hadron physics is a primary scientific funding priority of the nuclear physics community in the USA and the high-energy/hadron-physics community in Germany. A very high priority budget item in the Nuclear Science Advisory Committee's Long-Range (five-year) plan is an upgrade of the Continuous Electron Beam Accelerator Facility (CEBAF) at TNJAF and the construction of a new experimental hall, which will focus on searching for "exotic hadrons;" i.e., those states expected in QCD but impossible to build in the naive constituent quark model.

One thing that must be understood: the constituent quark model is an *ancient* tool. It's quantitative predictions are generally wrong and when they are right it is for the wrong reasons. Its only use in modern hadron physics is to provide illustrations. Contemporary theoretical analysis in Hadron physics focuses increasingly on developing a Poincaré covariant approach and that can be founded on a calculation of the Schwinger functions in QCD.

Contents

Chapter 1

Green Function in Relativistic Quantum Mechanics

1.1 Introduction

The study of relativistic quantum mechanics can be pursued in such a way as to make the Green function of the Dirac operator the keystone.

More to come.

1.2 Minkowski Space Conventions

This will probably be moved to an earlier point.

In this part of our text we will use the Minkowski metric conventions of Ref. [BD64]. Later we will employ a Euclidean metric because that is most useful and appropriate for nonperturbative calculations.

Four Vectors

Normal spacetime coordinates are denoted by a *contravariant* four-vector:

$$x^\mu := (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)\,. \tag{1.2.1}$$

Throughout: $c = 1 = \hbar$, and the conversion between length and energy is just:

$$1\,\mathrm{fm} = 1/(0.197327\,\mathrm{GeV}) = 5.06773\,\mathrm{GeV}^{-1}\,. \tag{1.2.2}$$

The *covariant* four-vector is obtained by changing the sign of the spatial components of the contravariant vector:

$$x_\mu := (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z) = g_{\mu\nu}x^\nu\,, \tag{1.2.3}$$

where the metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \tag{1.2.4}$$

The contracted product of two four-vectors is

$$(a, b) := g_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu \, ; \tag{1.2.5}$$

i.e, a contracted product of a covariant and contravariant four-vector. The Poincaré-invariant length of any vector is $x^2 := (x, x) = t^2 - \vec{x}^2$.

Momentum vectors are similarly defined:

$$p^\mu = (E, p_x, p_y, p_z) = (E, \vec{p}) \tag{1.2.6}$$

and

$$(p, k) = p_\mu k^\mu = E_p E_k - \vec{p} \cdot \vec{k} \, . \tag{1.2.7}$$

Likewise,

$$(x, p) = tE - \vec{x} \cdot \vec{p} . \tag{1.2.8}$$

The momentum *operator*

$$\boldsymbol{p}^\mu := i \frac{\partial}{\partial x_\mu} = (i \frac{\partial}{\partial t}, \frac{1}{i} \vec{\nabla}) =: i \nabla^\mu \tag{1.2.9}$$

transforms as a <u>contravariant</u> four-vector, and we denote the four-vector analogue of the Laplacian as

$$\partial^2 := -\boldsymbol{p}^\mu \boldsymbol{p}_\mu = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} \, . \tag{1.2.10}$$

The contravariant four-vector associated with the electromagnetic field is

$$A^\mu(x) = (\Phi(x), \vec{A}(x)) \tag{1.2.11}$$

with the electric and magnetic field strengths obtained from

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\mu \, ; \tag{1.2.12}$$

for example,

$$\vec{E}_i = F^{0i}\,; \text{ i.e. } \vec{E} = -\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\,. \tag{1.2.13}$$

Similarly, $B^i = \epsilon^{ijk}F^{jk}$, $j,k = 1,2,3$. Analogous definitions hold in QCD for the chromomagnetic field strengths.

Dirac Matrices

The Dirac matrices are indispensable in a manifestly Poincaré covariant description of particles with *spin*; i.e., intrinsic angular momentum, such as fermions with spin $\frac{1}{2}$.

The Dirac matrices are defined by the Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,, \tag{1.2.14}$$

where an identity matrix is implicit on the r.h.s., and one common 4×4 representation is [each entry represents a 2×2 matrix]

$$\gamma^0 = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix},\ \vec{\gamma} = \begin{bmatrix} \mathbf{0} & \vec{\boldsymbol{\sigma}} \\ -\vec{\boldsymbol{\sigma}} & \mathbf{0} \end{bmatrix}, \tag{1.2.15}$$

where $\vec{\sigma}$ are the usual Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1.2.16}$$

and $\mathbf{1} = \mathrm{diag}[1,1]$. Clearly: $\gamma_0^\dagger = \gamma_0$; and $\vec{\gamma}^\dagger = -\vec{\gamma}$. NB. These properties are not specific to this representation; e.g., $\gamma^1\gamma^1 = -\mathbf{1}_{4\times 4}$ for any representation of the Clifford algebra.

In discussing spin, two combinations of Dirac matrices frequently appear:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,,\ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5\,, \tag{1.2.17}$$

and I note that

$$\gamma^5\,\sigma^{\mu\nu} = \frac{i}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma}\,, \tag{1.2.18}$$

with $\epsilon^{\mu\nu\rho\sigma}$ the completely antisymmetric Lèvi-Civita tensor: $\epsilon^{0123} = +1$, $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$. In the representation introduced above,

$$\gamma^5 = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}. \tag{1.2.19}$$

Furthermore

$$\{\gamma_5, \gamma^\mu\} = 0\,,\ \gamma_5^\dagger = \gamma_5 \tag{1.2.20}$$

The Dirac matrix γ_5 plays a special role in the discussion of parity and chiral symmetry, two key aspects of the Standard Model.

The "slash" notation is a frequently used shorthand:

$$\gamma^\mu A_\mu =: \not{A} = \gamma^0 A^0 - \vec{\gamma}\cdot\vec{A}\,, \tag{1.2.21}$$

$$\gamma^\mu p_\mu =: \not{p} = \gamma^0 E - \vec{\gamma}\cdot\vec{p}\,, \tag{1.2.22}$$

$$\gamma^\mu \boldsymbol{p}_\mu =: i\not{\nabla} \equiv i\not{\partial} = i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} = i\gamma^\mu\frac{\partial}{\partial x^\mu}\,. \tag{1.2.23}$$

The following identities are important in evaluating the cross-sections for decay and scattering processes:

$$\mathrm{tr}\,\gamma_5 = 0\,, \tag{1.2.24}$$

$$\mathrm{tr}\,\mathbf{1} = 4\,, \tag{1.2.25}$$

$$\mathrm{tr}\,\not{a}\not{b} = 4(a,b)\,, \tag{1.2.26}$$

$$\mathrm{tr}\,\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4\left[(a_1,a_2)(a_3,a_4) - (a_1,a_3)(a_2,a_4) + (a_1,a_4)(a_2,a_3)\right], \tag{1.2.27}$$

$$\mathrm{tr}\,\not{a}_1\ldots\not{a}_n = 0\,,\ \text{for n odd}\,, \tag{1.2.28}$$

$$\mathrm{tr}\,\gamma_5\not{a}\not{b} = 0\,, \tag{1.2.29}$$

$$\mathrm{tr}\,\gamma_5\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4i\epsilon_{\alpha\beta\gamma\delta}\,a^\alpha b^\beta c^\gamma d^\delta\,, \tag{1.2.30}$$

$$\gamma_\mu\not{a}\gamma^\mu = -2\not{a}\,, \tag{1.2.31}$$

$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a,b)\,, \tag{1.2.32}$$

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}\,, \tag{1.2.33}$$

$$\tag{1.2.34}$$

They can all be derived using the fact that Dirac matrices satisfy the Clifford algebra. For example (remember $\mathrm{tr}\,AB = \mathrm{tr}\,BA$, where "tr" is the matrix trace operation):

$$\begin{aligned}\mathrm{tr}\,\not{a}\not{b} &= a_\mu b_\nu \mathrm{tr}\,\gamma^\mu\gamma^\nu \\ &= a_\mu b_\nu\,\frac{1}{2}\mathrm{tr}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right] \\ &= a_\mu b_\nu\,\frac{1}{2}\mathrm{tr}\left[2g^{\mu\nu}\mathbf{1}\right]\end{aligned}$$

$$= \quad a_\mu b_\nu \, \frac{1}{2} \, 2 g^{\mu\nu} \, 4 = 4(a,b) \, . \tag{1.2.35}$$

1.3 Dirac Equation

The unification of special relativity (Poincaré covariance) and quantum mechanics took some time. Even today many questions remain as to a practical implementation of an Hamiltonian formulation of the relativistic quantum mechanics of interacting systems. The Poincaré group has ten generators: the six associated with the Lorentz transformations – rotations and boosts – and the four associated with translations. Quantum mechanics describes the time evolution of a system with interactions, and that evolution is generated by the Hamiltonian. However, if the theory is formulated with an interacting Hamiltonian then boosts will almost always fail to commute with the Hamiltonian and thus the state vector calculated in one momentum frame will not be kinematically related to the state in another frame. That makes a new calculation necessary in every frame and hence the discussion of scattering, which takes a state of momentum p to another state with momentum p', is problematic. (See, e.g., Ref. [KP91].)

The Dirac equation provides the starting point for a Lagrangian formulation of the quantum field theory for fermions interacting via gauge boson exchange. For a free fermion

$$[i\partial\!\!\!/ - m] \, \psi = 0 \, , \tag{1.3.1}$$

where $\psi(x)$ is the fermion's "spinor" – a four component column vector, while in the presence of an external electromagnetic field the fermion's wave function obeys

$$[i\partial\!\!\!/ - eA\!\!\!/ - m] \, \psi = 0 \, , \tag{1.3.2}$$

which is obtained, as usual, via "minimal substitution:" $\boldsymbol{p}^\mu \to \boldsymbol{p}^\mu - eA^\mu$ in Eq. (1.3.1). These equations have a manifestly covariant appearance but proving their covariance requires the development of a representation of Lorentz transformations on spinors and Dirac matrices:

$$\psi'(x') \quad = \quad S(\Lambda) \, \psi(x) \, , \tag{1.3.3}$$

$$\Lambda^\nu_{\ \mu} \gamma^\mu \quad = \quad S^{-1}(\Lambda) \gamma^\nu S(\Lambda) \, , \tag{1.3.4}$$

$$S(\Lambda) \quad = \quad \exp[-\frac{i}{2} \, \sigma_{\mu\nu} \, \omega^{\mu\nu}] \, , \tag{1.3.5}$$

where $\omega^{\mu\nu}$ are the parameters characterising the particular Lorentz transformation. (Details can be found in the early chapters of Refs. [BD64; IZ80].)

Free Particle Solutions

As usual, to obtain an explicit form for the free-particle solutions one substitutes a plane wave and finds a constraint on the wave number. In this case there are two qualitatively different types of solution, corresponding to positive and negative energy. (An appreciation of the physical reality of the negative energy solutions led to the prediction of the existence of antiparticles.) One inserts

$$\psi^{(+)} = \mathrm{e}^{-i(k,x)}\, u(k)\,,\ \psi^{(-)} = \mathrm{e}^{+i(k,x)}\, v(k)\,, \tag{1.3.6}$$

into Eq. (1.3.1) and obtains

$$(\not{k} - m)\, u(k) = 0\,,\ (\not{k} + m)\, v(k) = 0\,. \tag{1.3.7}$$

Assuming that the particle's mass in nonzero then working in the rest frame yields

$$(\gamma^0 - \mathbf{1})\, u(m, \vec{0}) = 0\,,\ (\gamma^0 + \mathbf{1})\, v(m, \vec{0}) = 0\,. \tag{1.3.8}$$

There are clearly (remember the form of γ^0) two linearly-independent solutions of each equation:

$$u^{(1)}(m, \vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,,\ u^{(2)}(m, \vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\,, \tag{1.3.9}$$

$$v^{(1)}(m, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,,\ v^{(2)}(m, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{1.3.10}$$

The solution in an arbitrary frame can be obtained simply via a Lorentz boost but it is even simpler to observe that

$$(\not{k} - m)\,(\not{k} + m) = k^2 - m^2 = 0\,, \tag{1.3.11}$$

with the last equality valid when the particles are on shell, so that the solutions for arbitrary k^μ are: for positive energy ($E > 0$),

$$u^{(\alpha)}(k) = \frac{\not{k} + m}{\sqrt{2m(m+E)}}\, u^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \left(\frac{E+m}{2m}\right)^{1/2} & \phi^\alpha(m,\vec{0}) \\ \frac{\sigma\cdot k}{\sqrt{2m(m+E)}} & \phi^\alpha(m,\vec{0}) \end{pmatrix}, \tag{1.3.12}$$

with the two-component spinors, obviously to be identified with the fermion's spin in the rest frame (the only frame in which spin has its naive meaning)

$$\phi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \phi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \tag{1.3.13}$$

and, for negative energy,

$$v^{(\alpha)}(k) = \frac{-\not{k} + m}{\sqrt{2m(m+E)}}\, v^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \frac{\sigma\cdot k}{\sqrt{2m(m+E)}} & \chi^\alpha(m,\vec{0}) \\ \left(\frac{E+m}{2m}\right)^{1/2} & \chi^\alpha(m,\vec{0}) \end{pmatrix}, \tag{1.3.14}$$

with $\chi^{(\alpha)}$ obvious analogues of $\phi^{(\alpha)}$ in Eq. (1.3.13). (This approach works because it is clear that there are two, and only two, linearly-independent solutions of the momentum space free-fermion Dirac equations, Eqs. (1.3.7), and, for the homogeneous equations, any two covariant solutions with the correct limit in the rest-frame must give the correct boosted form.)

In quantum field theory, as in quantum mechanics, one needs a conjugate state to define an inner product. For fermions in Minkowski space that conjugate is $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, which satisfies

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0\,. \tag{1.3.15}$$

This equation yields the following free particle spinors in momentum space

$$\bar{u}^{(\alpha)}(k) = \bar{u}^{(\alpha)}(m,\vec{0})\, \frac{\not{k} + m}{\sqrt{2m(m+E)}} \tag{1.3.16}$$

$$\bar{v}^{(\alpha)}(k) = \bar{v}^{(\alpha)}(m,\vec{0})\, \frac{-\not{k} + m}{\sqrt{2m(m+E)}}\,, \tag{1.3.17}$$

where we have used $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, which relation is easily derived and is particularly important in the discussion of intrinsic parity; i.e., the transformation properties of particles and antiparticles under space reflections (an improper Lorentz transformation).

The momentum space free-particle spinors are orthonormalised

$$\begin{array}{ll} \bar{u}^{(\alpha)}(k)\, u^{(\beta)}(k) = \delta_{\alpha\beta} & \bar{u}^{(\alpha)}(k)\, v^{(\beta)}(k) = 0 \\ \bar{v}^{(\alpha)}(k)\, v^{(\beta)}(k) = -\delta_{\alpha\beta} & \bar{v}^{(\alpha)}(k)\, u^{(\beta)}(k) = 0 \end{array} . \tag{1.3.18}$$

It is now possible to construct positive and negative energy projection operators. Consider

$$\Lambda_+(k) \;:=\; \sum_{\alpha=1,2} u^{(\alpha)}(k) \otimes \bar{u}^{(\alpha)}(k)\,. \tag{1.3.19}$$

It is plain from the orthonormality relations, Eqs. (1.3.18), that $\Lambda_+(k)$ projects onto positive energy spinors in momentum space; i.e.,

$$\Lambda_+(k)\, u^{(\alpha)}(k) = u^{(\alpha)}(k)\,,\ \Lambda_+(k)\, v^{(\alpha)}(k) = 0\,. \tag{1.3.20}$$

Now, since

$$\sum_{\alpha=1,2} u^{(\alpha)}(m,\vec{0}) \otimes \bar{u}^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{\mathbf{1}+\gamma^0}{2}\,, \tag{1.3.21}$$

then

$$\Lambda_+(k) \;=\; \frac{1}{2m(m+E)}\,(\not{k}+m)\,\frac{\mathbf{1}+\gamma^0}{2}\,(\not{k}+m)\,. \tag{1.3.22}$$

Noting that for $k^2 = m^2$; i.e., on shell,

$$(\not{k}+m)\,\gamma^0\,(\not{k}+m) \;=\; 2E\,(\not{k}+m)\,, \tag{1.3.23}$$

$$(\not{k}+m)\,(\not{k}+m) \;=\; 2m\,(\not{k}+m)\,, \tag{1.3.24}$$

one finally arrives at the simple closed form:

$$\Lambda_+(k) = \frac{\not{k}+m}{2\,m}\,. \tag{1.3.25}$$

The negative energy projection operator is

$$\Lambda_-(k) \;:=\; -\sum_{\alpha=1,2} v^{(\alpha)}(k) \otimes \bar{v}^{(\alpha)}(k) = \frac{-\not{k}+m}{2\,m}\,. \tag{1.3.26}$$

The projection operators have the following characteristic and important properties:

$$\Lambda_\pm^2(k) = \Lambda_\pm(k)\,, \tag{1.3.27}$$

$$\mathrm{tr}\,\Lambda_\pm(k) = 2\,, \tag{1.3.28}$$

$$\Lambda_+(k) + \Lambda_-(k) = \mathbf{1}\,. \tag{1.3.29}$$

1.4 Green Functions

The Dirac equation is a partial differential equation. A general method for solving such equations is to use a Green function, which is the inverse of the differential operator that appears in the equation. The analogy with matrix equations is obvious and can be exploited heuristically.

Equation (1.3.2):

$$[i\partial\!\!\!/_x - eA\!\!\!/(x) - m]\,\psi(x) = 0\,, \tag{1.4.1}$$

yields the wave function for a fermion in an external electromagnetic field. Consider the operator obtained as a solution of the following equation

$$[i\partial\!\!\!/_{x'} - eA\!\!\!/(x') - m]\,S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \tag{1.4.2}$$

It is immediately apparent that if, at a given spacetime point x, $\psi(x)$ is a solution of Eq. (1.4.1), then

$$\psi(x') := \int d^4x\, S(x',x)\,\psi(x) \tag{1.4.3}$$

is a solution of

$$[i\partial\!\!\!/_{x'} - eA\!\!\!/(x') - m]\,\psi(x') = 0\,; \tag{1.4.4}$$

i.e., $S(x',x)$ has propagated the solution at x to the point x'.

This effect is equivalent to the application of Huygen's principle in wave mechanics: if the wave function at x, $\psi(x)$, is known, then the wave function at x' is obtained by considering $\psi(x)$ as a source of spherical waves that propagate outward from x. The amplitude of the wave at x' is proportional to the amplitude of the original wave, $\psi(x)$, and the constant of proportionality is the propagator (Green function), $S(x',x)$. The total amplitude of the wave at x' is the sum over all the points on the wavefront; i.e., Eq. (1.4.3).

This approach is practical because all physically reasonable external fields can only be nonzero on a compact subdomain of spacetime. Therefore the solution of the complete equation is transformed into solving for the Green function, which can then be used to propagate the free-particle solution, already found, to arbitrary spacetime points. However, obtaining the exact form of $S(x', x)$ is impossible for all but the simplest cases (see, e.g., Ref. [DR85; DR86]).

Free-Fermion Propagator

In the absence of an external field Eq. (1.4.2) becomes

$$[i\partial\!\!\!/_{x'} - m]\, S(x', x) = \mathbf{1}\, \delta^4(x' - x)\,. \tag{1.4.5}$$

Assume a solution of the form:

$$S_0(x', x) = S_0(x' - x) = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-i(p, x'-x)}\, S_0(p)\,, \tag{1.4.6}$$

so that substituting yields

$$(p\!\!\!/ - m)\, S_0(p) = \mathbf{1}\,;\ \text{i.e., } S_0(p) = \frac{p\!\!\!/ + m}{p^2 - m^2}\,. \tag{1.4.7}$$

To obtain the result in configuration space one must adopt a prescription for handling the on-shell singularities in $S(p)$, and that convention is tied to the boundary conditions applied to Eq. (1.4.5). An obvious and physically sensible definition of the Green function is that it should propagate positive-energy-fermions and -antifermions forward in time but not backwards in time, and vice versa for negative energy states.

As we have already seen, the wave function for a positive energy free-fermion is

$$\psi^{(+)}(x) = u(p)\, \mathrm{e}^{-i(p,x)}\,. \tag{1.4.8}$$

The wave function for a positive-energy antifermion is the charge-conjugate of the negative-energy fermion solution:

$$\psi_c^{(+)}(x) = C\, \gamma^0 \left(v(p)\, \mathrm{e}^{i(p,x)} \right)^* = C\, \bar{v}(p)^{\mathrm{T}}\, \mathrm{e}^{-i(p,x)}\,, \tag{1.4.9}$$

where $C = i\gamma^2\gamma^0$ and $(\cdot)^{\mathrm{T}}$ denotes matrix transpose.* It is thus evident

*This defines the operation of charge conjugation. NB. The form of C introduced here is appropriate to the Dirac matrix representation introduced above. Nevertheless, the unitary equivalence of Clifford algebra representations ensures that generality is not

that our physically sensible $S_0(x'-x)$ must only contain positive-frequency components for $x'_0 - x_0 > 0$.

One can ensure this via a small modification of the denominator of Eq. (1.4.7):

$$S_0(p) = \frac{\not{p} + m}{p^2 - m^2} \to \frac{\not{p} + m}{p^2 - m^2 + i\eta}\,, \tag{1.4.10}$$

with $\eta \to 0^+$ at the end of all calculations. Inserting this form in Eq. (1.4.6) is equivalent to evaluating the p^0 integral by employing a contour in the complex-p^0 that is below the real-p^0 axis for $p^0 < 0$, and above it for $p^0 > 0$. This prescription defines the **Feynman** propagator.

To be explicit:

$$\begin{aligned} S_0(x'-x) = \int \frac{d^3p}{(2\pi)^3}\, \mathrm{e}^{i\vec{p}\cdot(\vec{x}'-\vec{x})}\, \frac{1}{2\,\omega(\vec{p})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \\ \times \left[\mathrm{e}^{-ip^0(x'_0-x_0)}\, \frac{\not{p} + m}{p^0 - \omega(\vec{p}) + i\eta} - \mathrm{e}^{-ip^0(x'_0-x_0)}\, \frac{\not{p} + m}{p^0 + \omega(\vec{p}) - i\eta} \right], \end{aligned} \tag{1.4.11}$$

where the energy $\omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$. The integrals are easily evaluated using standard techniques of complex analysis, in particular, Cauchy's Theorem.

Focusing on the first term of the sum inside the square brackets, it is apparent that the integrand has a pole in the fourth quadrant of the complex p^0-plane. For $x'_0 - x_0 > 0$ we can evaluate the p^0 integral by considering a contour closed by a semicircle of radius $R \to \infty$ in the lower half of the complex p^0-plane: the integrand vanishes exponentially along that arc, where $p^0 = -iy$, $y > 0$, because $(-i)\,(-iy)\,(x'_0 - x_0) = -y\,(x'_0 - x_0) < 0$. The closed contour is oriented clockwise so that

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x'_0-x_0)}\, \frac{\not{p} + m}{p^0 - \omega(\vec{p}) + i\eta^+} \\ &= \quad (-)\, i\ \mathrm{e}^{-ip^0(x'_0-x_0)} (\not{p} + m)\Big|_{p^0=\omega(\vec{p})-i\eta^+} \\ &= \quad -i\, \mathrm{e}^{-i\omega(\vec{p})(x'_0-x_0)}\, (\gamma^0\omega(\vec{p}) - \gamma\cdot\vec{p} + m) \\ &= \quad -i\, \mathrm{e}^{-i\omega(\vec{p})(x'_0-x_0)}\, 2m\,\Lambda_+(p)\,. \end{aligned} \tag{1.4.12}$$

lost [BD64].

For $x_0' - x_0 < 0$ the contour must be closed in the upper half plane but therein the integrand is analytic and hence the result is zero. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x_0)}\, \frac{\not{p}+m}{p^0-\omega(\vec{p})+i\eta^+} \\ = \quad -i\,\theta(x_0'-x_0)\,\mathrm{e}^{-i\omega(\vec{p})(x_0'-x)}\,2m\,\Lambda_+(\vec{p})\,. \end{aligned} \tag{1.4.13}$$

Observe that the projection operator is only a function of $\vec{p}$ because $p^0 = \omega(\vec{p})$.

Consider now the second term in the brackets. The integrand has a pole in the second quadrant. For $x_0' - x_0 > 0$ the contour must be closed in the lower half plane, the pole is avoided and hence the integral is zero. However, for $x_0' - x_0 < 0$ the contour should be closed in the upper half plane, where the integrand is obviously zero on the arc at infinity. The contour is oriented anticlockwise so that

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x_0)}\, \frac{\not{p}+m}{p^0+\omega(\vec{p})-i\eta^+} \\ &= \quad i\,\theta(x_0-x_0')\;\mathrm{e}^{-ip^0(x_0'-x)}(\not{p}+m)\Big|_{p^0=-\omega(\vec{p})+i\eta^+} \\ &= \quad i\,\theta(x_0-x_0')\,\mathrm{e}^{+i\omega(\vec{p})(x_0'-x)}\,(-\gamma^0\omega(\vec{p})-\gamma\cdot\vec{p}+m) \\ &= \quad i\,\theta(x_0-x_0')\,\mathrm{e}^{+i\omega(\vec{p})(x_0'-x)}\,2m\,\Lambda_-(-\vec{p})\,. \end{aligned} \tag{1.4.14}$$

Putting these results together, changing variables $\vec{p} \to -\vec{p}$ in Eq. (1.4.14), we have

$$\begin{aligned}S_0(x'-x) \quad = \quad -i\int \frac{d^3p}{(2\pi)^3}\,\frac{m}{\omega(\vec{p})}\,\Big[\theta(x_0'-x_0)\,\mathrm{e}^{-i(\tilde{p},x'-x)}\,\Lambda_+(\vec{p}) \\ +\,\theta(x_0-x_0')\,\mathrm{e}^{i(\tilde{p},x'-x)}\Lambda_-(\vec{p})\Big]\,,\end{aligned}$$

$[(\tilde{p}^\mu) = (\omega(\vec{p}),\vec{p})]$ which, using Eqs. (1.3.20) and its obvious analogue for $\Lambda_-(\vec{p})$, manifestly propagates positive-energy solutions forward in time and negative energy solutions backward in time, and hence satisfies the physical requirement stipulated above.

Another useful representation is obtained merely by observing that

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, \frac{\not{p}+m}{p^0-\omega(\vec{p})+i\eta^+} \\ &= \quad \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, \frac{\gamma^0\omega(\vec{p})-\vec{\gamma}\cdot\vec{p}+m}{p^0-\omega(\vec{p})+i\eta^+}\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, 2m\Lambda_+(\vec{p})\, \frac{1}{p^0-\omega(\vec{p})+i\eta^+}\,, \quad (1.4.15)$$

and similarly

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, \frac{\not{p}+m}{p^0+\omega(\vec{p})-i\eta^+} \\
&= \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, \frac{-\gamma^0\omega(\vec{p})-\vec{\gamma}\cdot\vec{p}+m}{p^0+\omega(\vec{p})-i\eta^+} \\
&= \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x)}\, 2m\Lambda_-(-\vec{p})\, \frac{1}{p^0+\omega(\vec{p})-i\eta^+}\,, \quad (1.4.16)
\end{aligned}$$

so that Eq. (1.4.11) can be rewritten

$$\begin{aligned}
S_0(x'-x) \;=\; & \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-i(p,x'-x)}\, \frac{m}{\omega(\vec{p})} \left[\Lambda_+(\vec{p})\, \frac{1}{p^0-\omega(\vec{p})+i\eta}\right. \\
& \left. -\Lambda_-(-\vec{p})\, \frac{1}{p^0+\omega(\vec{p})-i\eta}\right].
\end{aligned}$$

The utility of this representation is that it provides a single Fourier amplitude for the free-fermion Green function; i.e., an alternative to Eq. (1.4.10):

$$S_0(p) = \frac{m}{\omega(\vec{p})} \left[\Lambda_+(\vec{p})\, \frac{1}{p^0-\omega(\vec{p})+i\eta} - \Lambda_-(-\vec{p})\, \frac{1}{p^0+\omega(\vec{p})-i\eta}\right], \quad (1.4.17)$$

which is indispensable in making a connection between covariant perturbation theory and time-ordered perturbation theory: the second term generates the Z-diagrams in loop integrals.

Green Function for the Interacting Theory

We now return to Eq. (1.4.2):

$$[i\not{\partial}_{x'} - e\not{A}(x') - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x)\,, \quad (1.4.18)$$

which defines the Green function for a fermion in an external electromagnetic field. As mentioned, a closed form solution of this equation is impossible in all but the simplest field configurations. Is there, nevertheless, a way to construct an approximate solution that can be systematically improved?

To achieve that one rewrites the equation:

$$[i\not{\partial}_{x'} - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x) + e\not{A}(x')\, S(x',x)\,, \quad (1.4.19)$$

which, as is easily seen by substitution, is solved by

$$\begin{aligned} & S(x',x) \\ &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not{A}(y)\, S(y,x) \qquad (1.4.20) \\ &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not{A}(y)\, S_0(y-x) \\ &\quad + e^2\int d^4y_1 \int d^4y_2\, S_0(x'-y_1)\not{A}(y_1)\, S_0(y_1-y_2)\not{A}(y_2)\, S_0(y_2-x) \\ &\quad + \ldots \qquad (1.4.21) \end{aligned}$$

This perturbative expansion of the full propagator in terms of the free propagator provides an archetype for perturbation theory in quantum field theory. One obvious application is the scattering of an electron/positron by a Coulomb field, which is an example explored in Sec. 2.5.3 of Ref. [IZ80]. Equation (1.4.21) is a first example of a Dyson-Schwinger equation [RW94].

This Green function has the following interpretation:

(1) It creates a positive energy fermion (antifermion) at spacetime point x;
(2) Propagates the fermion to spacetime point x'; i.e., forward in time;
(3) Annihilates this fermion at x'.

The process can equally well be viewed as

(1) The creation of a negative energy antifermion (fermion) at spacetime point x';
(2) Propagation of the antifermion to the spacetime point x; i.e., backward in time;
(3) Annihilation of this antifermion at x.

Other propagators have similar interpretations.

1.5 Exercises

(1) Prove these relations for on-shell fermions:

$$\begin{aligned} (\not{k}+m)\,\gamma^0\,(\not{k}+m) &= 2E\,(\not{k}+m)\,, \\ (\not{k}+m)\,(\not{k}+m) &= 2m\,(\not{k}+m)\,. \end{aligned}$$

(2) Obtain the Feynman propagator for the free-field Klein Gordon equation:

$$(\partial_x^2 + m^2)\phi(x) = 0\,, \tag{1.5.1}$$

in forms analogous to Eqs. (1.4.15), (1.4.17).

Chapter 2
Functional Integrals

2.1 Path Integral in Quantum Mechanics

Local gauge theories are the keystone of contemporary hadron and high-energy physics. Such theories are difficult to quantise because the gauge dependence is an extra non-dynamical degree of freedom that must be dealt with. The modern approach is to quantise the theories using the method of functional integrals, attributed to Feynman and Kac. References [IZ80; PT84] provide useful overviews of the technique, which replaces canonical second-quantisation.

It is useful to motivate the functional integral formulation by reviewing the path integral representation of the transition amplitude in quantum mechanics. Beginning with a state $(q_1, q_2, \ldots, q_N)$ at time t, the probability that one will obtain the state $(q'_1, q'_2, \ldots, q'_N)$ at time t' is given by (remember, the time evolution operator in quantum mechanics is $\exp[-iHt]$, where H is the system's Hamiltonian):

$$\begin{aligned}
&\langle q'_1, q'_2, \ldots, q'_N; t' | q_1, q_2, \ldots, q_N; t \rangle \\
&= \lim_{n\to\infty} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} dq_\alpha(t_i) \int \prod_{i=1}^{n+1} \frac{dp_\alpha(t_i)}{2\pi} \\
&\quad \times \exp\left[i \sum_{j=1}^{n+1} \left\{ p_\alpha(t_j)[q_\alpha(t_j) - q_\alpha(t_{j-1})] - \epsilon H(p(t_j), \frac{q_{t_j} + q_{t_j-1}}{2}) \right\} \right],
\end{aligned} \tag{2.1.1}$$

where $t_j = t + j\epsilon$, $\epsilon = (t'-t)/(n+1)$, $t_0 = t$, $t_{n+1} = t'$. A compact notation

is commonly introduced to represent this expression:

$$\langle q't'|qt\rangle^J = \int [dq] \int [dp]\ \mathrm{e}^{i\int_t^{t'} d\tau\,[p(\tau)\dot{q}(\tau)-H(\tau)+J(\tau)q(\tau)]} \tag{2.1.2}$$

where $J(t)$ is a classical external "source." NB. The $J = 0$ exponent is nothing but the Lagrangian.

Recall that in Heisenberg's formulation of quantum mechanics it is the operators that evolve in time and not the state vectors, whose values are fixed at a given initial time. Using the previous formulae it is a simple matter to prove that the time ordered product of n Heisenberg position operators can be expressed as

$$\begin{aligned}&\langle q't'|T\{Q(t_1)\dots Q(t_n)\}|qt\rangle \\ &\quad = \int [dq] \int [dp]\, q(t_1)\, q(t_2)\dots q(t_n)\ \mathrm{e}^{i\int_t^{t'} d\tau\,[p(\tau)\dot{q}(\tau)-H(\tau)]}\end{aligned} \tag{2.1.3}$$

NB. The time ordered product ensures that the operators appear in chronological order, right to left.

Consider a source that "switches on" at t_i and "switches off" at t_f, $t < t_i < t_f < t'$, then

$$\langle q't'|qt\rangle^J = \int dq_i dq_f\, \langle q't'|q_f t_f\rangle\, \langle q_f t_f|q_i t_i\rangle^J\, \langle q_i t_i|qt\rangle\,. \tag{2.1.4}$$

One can alternatively introduce a complete set of energy eigenstates to resolve the Hamiltonian and write

$$\langle q't'|qt\rangle \quad = \quad \sum_n \langle q'|\phi_n\rangle\, \mathrm{e}^{-iE_n(t'-t)}\, \langle \phi_n|q\rangle \tag{2.1.5}$$

$$\overset{\substack{t'\to-i\infty\\ t\to+i\infty}}{=} \quad \langle q'|\phi_0\rangle\, \mathrm{e}^{-iE_0(t'-t)}\, \langle \phi_0|q\rangle\,; \tag{2.1.6}$$

i.e., in either of these limits the transition amplitude is dominated by the ground state.

It now follows from Eqs. (2.1.4), (2.1.6) that

$$W[J] := \lim_{\substack{t'\to-i\infty\\ t\to+i\infty}} \frac{\langle q't'|qt\rangle}{\mathrm{e}^{-iE_0(t'-t)}\, \langle q'|\phi_0\rangle\, \langle \phi_0|q\rangle} \tag{2.1.7}$$

$$= \int dq_i dq_f\, \langle \phi_0|q_f t_f\rangle\, \langle q_f t_f|q_i t_i\rangle^J\, \langle q_i t_i|\phi_0\rangle \tag{2.1.8}$$

i.e., the ground-state to ground-state transition amplitude (survival probability) in the presence of the external source J. From this it will readily be apparent that, with $t_f > t_1 > t_2 > \ldots > t_m > t_i$,

$$\begin{aligned}\left.\frac{\delta^m W[J]}{\delta J(t_1)\,\ldots\,\delta J(t_1)}\right|_{J=0} \\ = \quad i^m \int dq_i dq_f \,\langle\phi_0|q_f t_f\rangle\,\langle q_f t_f|T\{Q(t_1)\ldots Q(t_n)\}|q_i t_i\rangle\,\langle q_i t_i|\phi_0\rangle,\end{aligned} \tag{2.1.9}$$

which is the ground state expectation value of a time ordered product of Heisenberg position operators. The analogues of these expectation values in quantum field theory are the Green functions.

The functional derivative introduce here: $\frac{\delta}{\delta J(t)}$, is defined analogously to the derivative of a function. It means: write $J(t) \to J(t) + \epsilon(t)$; expand the functional in $\epsilon(t)$; and identify the leading order coefficient in the expansion as the functional derivative. Thus for

$$H_n[J] = \int dt'\, J(t')^n\,, \tag{2.1.10}$$

$$\begin{aligned}\delta H_n[J] &= \delta \int dt'\, J(t')^n = \int dt'\,[J(t') + \epsilon(t')]^n - \int dt'\, J(t')^n \\ &= \int dt'\, n\, J(t')^{n-1}\,\epsilon(t') + [\ldots].\end{aligned} \tag{2.1.11}$$

So that taking the limit $\delta J(t') = \epsilon(t') \to \delta(t - t')$ one obtains

$$\frac{\delta H_n[J]}{\delta J(t)} = n\, J(t)^{n-1}\,. \tag{2.1.12}$$

The limit procedure just described corresponds to defining

$$\frac{\delta J(t)}{\delta J(t')} = \delta(t - t')\,. \tag{2.1.13}$$

This example, which is, in fact, the "product rule," makes plain the very close analogy between functional differentiation and the differentiation of functions so that, with a little care, the functional differentiation of complicated functionals is straightforward.

Another note is in order. The fact that the limiting values in Eqs. (2.1.6), (2.1.7) are imaginary numbers is a signal that mathematical rigour may

more easily be found in Euclidean space where $t \to -it^E$. Alternatively, at least in principle, the argument can be repeated and made rigorous by making the replacement $E_n \to E_n - i\eta$, with $\eta \to 0^+$. However, that expedient does not help in a direct calculation of $W[J]$, which is defined by a multidimensional integral that can only be evaluated via Monte-Carlo methods; i.e., probability sampling, a method that is only effective when the integrand is positive definite.

2.2 Scalar Quantum Field

The Euclidean functional integral is particularly well suited to a direct numerical evaluation via Monte-Carlo methods because it defines a probability measure. However, to make direct contact with the perturbation theory of canonical second-quantised quantum field theory, we begin with a discussion of the Minkowski space formulation.

Generating Functional

We will consider a scalar field $\phi(t,x)$, which is customary because it reduces the number of indices that must be carried through the calculation. Suppose that a large but compact domain of space is divided into N cubes of volume ϵ^3 and label each cube by an integer α. Define the coordinate and momentum via

$$q_\alpha(t) := \phi_\alpha(t) = \frac{1}{\epsilon^3}\int_{V_\alpha} d^3x\, \phi(t,x)\,,\ \dot{q}_\alpha(t) := \dot{\phi}_\alpha(t) = \frac{1}{\epsilon^3}\int_{V_\alpha} d^3x\, \frac{\partial\phi(t,x)}{\partial t}\,; \tag{2.2.1}$$

i.e., as the spatial averages over the cube denoted by α.

The classical dynamics of the field ϕ is described by a Lagrangian:

$$\mathrm{L}(t) = \int d^3x\, L(t,x) \to \sum_{\alpha=1}^{N} \epsilon^3\, L_\alpha(\dot{\phi}_\alpha(t), \phi_\alpha(t), \phi_{\alpha\pm s}(t))\,, \tag{2.2.2}$$

where the dependence on $\phi_{\alpha\pm s}(t)$; i.e., the coordinates in the neighbouring cells, is necessary in order to express spatial derivatives in the Lagrangian density, $L(x)$.

With the canonical conjugate momentum defined as in a classical field theory

$$p_\alpha(t) := \frac{\partial L}{\partial\dot{\phi}_\alpha(t)} = \epsilon^3\frac{\partial L_\alpha}{\partial\dot{\phi}_\alpha(t)} =: \epsilon^3\pi_\alpha(t)\,, \tag{2.2.3}$$

the Hamiltonian is obtained as

$$\mathrm{H} = \sum_\alpha p_\alpha(t)\,\dot{q}_\alpha(t) - L(t) \quad =: \quad \sum_\alpha \epsilon^3\, H_\alpha\,, \tag{2.2.4}$$

$$H_\alpha(\pi_\alpha(t), \phi_\alpha(t), \phi_{\alpha\pm s}(t)) \quad = \quad \pi_\alpha(t)\,\dot{\phi}_\alpha(t) - L_\alpha\,. \tag{2.2.5}$$

The field theoretical equivalent of the quantum mechanical transition amplitude, Eq. (2.1.1), can now be written

$$\begin{aligned}
&\int [\mathcal{D}\phi] \int [\mathcal{D}\pi]\, \exp\left\{ i \int_t^{t'} d\tau \int d^3x \, \left[\pi(\tau,\vec{x}) \frac{\partial \phi(\tau,\vec{x})}{\partial \tau} - H(\tau,\vec{x}) \right] \right\} \\
&:= \lim_{n\to\infty, \epsilon\to 0^+} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} d\phi_\alpha(t_i) \int \prod_{i=1}^{n} \epsilon^3 \frac{d\pi_\alpha(t_i)}{2\pi} \\
&\quad \times \exp\left[i \sum_{j=1}^{n+1} \epsilon \sum_\alpha \epsilon^3 \left\{ \pi_\alpha(t_j) \frac{\phi_\alpha(t_j) - \phi_\alpha(t_{j-1})}{\epsilon} \right.\right. \\
&\quad \left.\left. - H_\alpha\left(\pi_\alpha(t_j), \frac{\phi_\alpha(t_j) + \phi_\alpha(t_{j-1})}{2}, \frac{\phi_{\alpha\pm s}(t_j) + \phi_{\alpha\pm s}(t_{j-1})}{2} \right) \right\} \right] ,
\end{aligned} \tag{2.2.6}$$

where, as classically, $\pi(t,\vec{x}) = \partial L(t,\vec{x})/\partial\dot{\phi}(t,\vec{x})$ and its average over a spacetime cube is just $\pi_\alpha(t)$. Equation (2.2.6) is the amplitude that describes a transition from an initial field configuration $\phi_\alpha(t_0) := \phi_\alpha(t)$ to a final configuration $\phi_\alpha(t_{n+1}) := \phi_\alpha(t')$.

In quantum field theory all physical quantities can be obtained from Green functions, which themselves are determined by vacuum-to-vacuum transition amplitudes calculated in the presence of classical external sources. The physical or interacting vacuum is the analogue of the true ground state in quantum mechanics. And, as in quantum mechanics, the fundamental quantity is the generating functional:

$$W[J] := \frac{1}{\mathcal{N}} \int [\mathcal{D}\phi][\mathcal{D}\pi]\; \mathrm{e}^{\,i\int d^4x\,[\pi(x)\dot{\phi}(x) - H(x) + \frac{1}{2}i\eta\phi^2(x) + J(x)\phi(x)]}, \tag{2.2.7}$$

where $\mathcal{N}$ is chosen so that $W[0] = 1$, and a real-time limit is implemented and made meaningful by adding the $\eta \to 0^+$ term. (NB. This subtracts a small imaginary piece from the mass.)

It is immediately apparent that

$$G(x_1, x_2, \ldots, x_n) := \frac{1}{i^n} \frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\bigg|_{J=0} \quad (2.2.8)$$

$$= \frac{\langle\tilde{0}|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle}, \quad (2.2.9)$$

where $|\tilde{0}\rangle$ is the physical vacuum. $G(x_1, x_2, \ldots, x_n)$ is the *complete* n-point Green function for the scalar quantum field theory; i.e., the vacuum expectation value of a time-ordered product of n field operators. The appearance of the word "complete" means that this Green function includes contributions from products of lower-order Green functions; i.e., disconnected diagrams.

The fact that the Green functions in a quantum field theory may be defined via Eq. (2.2.9) was first observed by Schwinger [SC51] and does not rely on the functional formula for $W[J]$, Eq. (2.2.7), for its proof. However, the functional formulism provides the simplest proof and, in addition, a concrete means of calculating the generating functional; i.e., via numerical simulations.

Connected Green Functions

It is useful to have a systematic procedure for the *a priori* elimination of disconnected parts from a n-point Green function because performing unnecessary work, as in the recalculation of $m < n$-point Green functions, is inefficient. A *connected* n-point Green function is given by

$$G_c(x_1, x_2, \ldots, x_n) = (-i)^{n-1} \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\bigg|_{J=0}, \quad (2.2.10)$$

where $Z[J]$, defined via

$$W[J] =: \exp\{iZ[J]\}, \quad (2.2.11)$$

is the generating functional for connected Green functions.

It is instructive to illustrate this for a simple case (recall that we have already proven the product rule for functional differentiation):

$$G_c(x_1, x_2) = (-i) \frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} = -\frac{\delta^2 \ln W[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0}$$

$$= -\frac{\delta}{\delta J(x_1)}\left[\frac{1}{W[J]}\frac{\delta W[J]}{\delta J(x_2)}\right]\bigg|_{J=0}$$

$$
\begin{aligned}
&= + \frac{1}{W^2[J]} \frac{\delta W[J]}{\delta J(x_1)} \frac{\delta W[J]}{\delta J(x_2)}\bigg|_{J=0} - \frac{1}{W[J]} \frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} \\
&= i\frac{\langle\tilde{0}|\hat{\phi}(x_1)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle}\, i\frac{\langle\tilde{0}|\hat{\phi}(x_2)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} - i^2\frac{\langle\tilde{0}|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \\
&= -G(x_1)\,G(x_2) + G(x_1,x_2)\,. \qquad (2.2.12)
\end{aligned}
$$

Generating Functional

The double functional integral employed above is cumbersome, especially since it involves the field variable's canonical conjugate momentum. Consider therefore a Hamiltonian density of the form

$$H(x) = \frac{1}{2}\pi^2(x) + f[\phi(x), \vec{\nabla}\phi(x)]\,. \qquad (2.2.13)$$

In this case Eq. (2.2.7) involves

$$
\begin{aligned}
&\int [\mathcal{D}\pi]\; \mathrm{e}^{i\int d^4x\,[-\frac{1}{2}\pi^2(x)+\pi(x)\dot{\phi}(x)]} \\
&= \mathrm{e}^{i\int d^4x\,[\dot{\phi}(x)]^2} \int [\mathcal{D}\pi]\; \mathrm{e}^{-\frac{i}{2}\int d^4x\,[\pi(x)-\dot{\phi}(x)]^2} \\
&= \mathrm{e}^{\{i\int d^4x\,[\dot{\phi}(x)]^2\}} \times \mathrm{N}\,, \qquad (2.2.14)
\end{aligned}
$$

where "N" is simply a constant. (This is an example of the only functional integral than can be evaluated exactly; i.e., the Gaussian functional integral.) Hence, with an Hamiltonian of the form in Eq. (2.2.13), the generating functional can be written

$$W[J] = \frac{\mathrm{N}}{\mathcal{N}} \int [\mathcal{D}\phi]\; \mathrm{e}^{i\int d^4x\,[L(x)+\frac{1}{2}i\eta\phi^2(x)+J(x)\phi(x)]} \qquad (2.2.15)$$

Recall now that the classical Lagrangian density for a scalar field is

$$
\begin{aligned}
L(x) &= L_0(x) + L_I(x)\,, \qquad (2.2.16) \\
L_0(x) &= \frac{1}{2}[\partial_\mu\phi(x)\,\partial^\mu\phi(x) - m^2\phi^2(x)]\,, \qquad (2.2.17)
\end{aligned}
$$

with $L_I(x)$ some functional of $\phi(x)$ that usually does not depend on any derivatives of the field. Hence the Hamiltonian for such a theory has the form in Eq. (2.2.13) and so Eq. (2.2.15) can be used to define the quantum field theory.

Quantum Field Theory for a Free Scalar Field

The interaction Lagrangian vanishes for a free scalar field so that the generating functional is, formally,

$$W_0[J] = \frac{1}{\mathcal{N}} \int [\mathcal{D}\phi] \; \mathrm{e}^{i \int d^4x \left[L_0(x) + i\eta\phi^2(x) \right] + J(x)\phi(x) \big]} \tag{2.2.18}$$

with $L_0(x)$ given in Eq. (2.2.17). The explicit meaning of Eq. (2.2.18) is

$$\begin{aligned} W_0[J] = & \lim_{\epsilon \to 0^+} \frac{1}{\mathcal{N}} \int \prod_\alpha d\phi_\alpha \\ & \times \exp \left\{ i \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \, \frac{1}{2} \, \phi_\alpha \, K_{\alpha\beta} \, \phi_\beta + \sum_\alpha \, \epsilon^4 \, J_\alpha \phi_\alpha \right\} , \end{aligned} \tag{2.2.19}$$

where α, β label spacetime hypercubes of volume ϵ^4 and $K_{\alpha\beta}$ is any matrix that satisfies

$$\lim_{\epsilon \to 0^+} K_{\alpha\beta} = [-\partial^2 - m^2 + i\eta] \, \delta^4(x - y) \, , \tag{2.2.20}$$

where $\alpha \overset{\epsilon \to 0^+}{\to} x$, $\beta \overset{\epsilon \to 0^+}{\to} y$ and $\sum_\alpha \epsilon^4 \overset{\epsilon \to 0^+}{\to} \int d^4x$; i.e., $K_{\alpha\beta}$ is any matrix whose continuum limit is the inverse of the Feynman propagator for a free scalar field. (NB. The fact that there are infinitely many such matrices provides the scope for "improving" lattice actions since one may choose $K_{\alpha\beta}$ wisely rather than for simplicity.)

Recall now that for matrices whose real part is positive definite

$$\begin{aligned} & \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i \; \exp\{ -\frac{1}{2} \sum_{i,j=1}^n \, x_i \, A_{ij} \, x_j + \sum_{i=1}^n \, b_i \, x_i \} \\ & = \quad \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \; \exp\{ \frac{1}{2} \sum_{i,j=1}^n \, b_i \, (A^{-1})_{ij} \, b_j \} \\ & = \quad \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \; \exp \left\{ \frac{1}{2} \boldsymbol{b}^{\mathrm{t}} A^{-1} \boldsymbol{b} \right\} . \end{aligned} \tag{2.2.21}$$

Hence Eq. (2.2.19) yields

$$W_0[J] = \lim_{\epsilon \to 0^+} \frac{1}{\mathcal{N}'} \, \frac{1}{\sqrt{\det A}} \; \exp\{ \frac{1}{2} \sum_\alpha \epsilon^4 \, \sum_\beta \epsilon^4 \, J_\alpha \, \frac{1}{i\epsilon^8} \, (K^{-1})_{\alpha\beta} \, J_\beta \} \, , \tag{2.2.22}$$

where, obviously, the matrix inverse is defined via

$$\sum_\gamma K_{\alpha\gamma}\,(K^{-1})_{\gamma\beta} = \delta_{\alpha\beta}\,. \tag{2.2.23}$$

Almost as obviously, consistency of limits requires

$$\lim_{\epsilon\to 0^+}\,\frac{1}{\epsilon^4}\,\delta_{\alpha\beta} = \delta^4(x-y)\,,\quad \lim_{\epsilon\to 0^+}\,\sum_\alpha \epsilon^4 = \int d^4x\,, \tag{2.2.24}$$

so that, with

$$\mathrm{O}(x,y) := \lim_{\epsilon\to 0^+}\,\frac{1}{\epsilon^8}(K^{-1})_{\alpha\beta}\,, \tag{2.2.25}$$

the continuum limit of Eq. (2.2.23) can be understood as follows:

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\sum_\gamma \epsilon^4\,K_{\alpha\gamma}\,\frac{1}{\epsilon^8}(K^{-1})_{\gamma\beta} = \lim_{\epsilon\to 0^+}\,\frac{1}{\epsilon^4}\,\delta_{\alpha\beta}\\
\Rightarrow\quad &\int d^4w\,[-\partial_x^2 - m^2 + i\eta]\,\delta^4(x-w)\,O(w,y) = \delta^4(x-y)\\
\therefore\quad &[-\partial_x^2 - m^2 + i\eta]\,O(x,y) = \delta^4(x-y)\,.
\end{aligned} \tag{2.2.26}$$

Hence $O(x,y) = \Delta_0(x-y)$; i.e., the Feynman propagator for a free scalar field:

$$\Delta_0(x-y) = \int \frac{d^4p}{(2\pi)^4}\mathrm{e}^{-i(q,x-y)}\frac{1}{q^2 - m^2 + i\eta}\,. \tag{2.2.27}$$

(NB. This makes plain the fundamental role of the "$i\eta^+$" prescription in Eq. (1.4.10): it ensures convergence of the expression defining the functional integral.)

Putting this all together, the continuum limit of Eq. (2.2.22) is

$$W[J] = \frac{1}{\hat{\mathcal{N}}}\,\mathrm{e}^{-\frac{i}{2}\int d^4x\int d^4y\,J(x)\,\Delta_0(x-y)\,J(y)}\,. \tag{2.2.28}$$

Scalar Field with Self-Interactions

A nonzero interaction Lagrangian, $L_I[\phi(x)]$, provides for a self-interacting scalar field theory (from here on we will usually omit the constant, nondynamical normalisation factor in writing the generating functional):

$$W[J] = \int [\mathcal{D}\phi]\exp\left\{i\int d^4x\,[L_0(x) + L_I(x) + J(x)\phi(x)]\right\}$$

$$
\begin{aligned}
&= \exp\left[i\int d^4x\, L_I\left(\frac{\delta}{i\delta J(x)}\right)\right] \\
&\quad \times \int [\mathcal{D}\phi] \exp\left\{i\int d^4x\,[L_0(x) + J(x)\phi(x)]\right\} \qquad (2.2.29)\\
&= \exp\left[i\int d^4x\, L_I\left(\frac{\delta}{i\delta J(x)}\right)\right] \\
&\quad \times \exp\left\{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\,\Delta_0(x-y)\,J(y)\right\}, \qquad (2.2.30)
\end{aligned}
$$

where

$$
\exp\left[i\int d^4x\, L_I\left(\frac{\delta}{i\delta J(x)}\right)\right] := \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[L_I\left(\frac{\delta}{i\delta J(x)}\right)\right]^n . \qquad (2.2.31)
$$

Equation (2.2.30) is the basis for a perturbative evaluation of all possible Green functions for the theory. As an example we will work through a first-order calculation of the complete 2-point Green function in the theory defined by

$$
L_I(x) = -\frac{\lambda}{4!}\,\phi^4(x)\,. \qquad (2.2.32)
$$

The generating functional yields

$$
\begin{aligned}
W[0] &= \left\{1 - i\frac{\lambda}{4!}\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4\right\} \qquad (2.2.33)\\
&\quad \times \exp\left\{-\frac{i}{2}\int d^4u \int d^4v\, J(u)\,\Delta_0(u-v)\,J(v)\right\}\Bigg|_{J=0} \\
&= 1 - i\frac{\lambda}{4!}\int d^4x\, 3\,[i\Delta_0(0)]^2\,. \qquad (2.2.34)
\end{aligned}
$$

The 2-point function is

$$
\begin{aligned}
\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)} &= -i\Delta_0(x_1-x_2) + i\,\frac{\lambda}{8}\int d^4x\,[i\Delta_0(0)]^2[i\Delta_0(x_1-x_2)] \\
&\quad + i\,\frac{\lambda}{2}\int d^4x\,[i\Delta_0(0)][i\Delta_0(x_1-x)][i\Delta_0(x-x_2)]\,. \qquad (2.2.35)
\end{aligned}
$$

Now using the definition, Eq. (2.2.9), and restoring the normalisation we

find

$$\begin{aligned} G(x_1,x_2) &= \frac{1}{i^2}\frac{1}{W[0]}\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)} \\ &= i\Delta_0(x_1-x_2) \\ &\quad -i\frac{\lambda}{2}\int d^4x\,[i\Delta_0(0)][i\Delta_0(x_1-x)][i\Delta_0(x-x_2)]\,, \end{aligned} \tag{2.2.36}$$

where we have used

$$\frac{a+\lambda b}{1+\lambda c} = a+\lambda(b-ac)+\mathrm{O}(\lambda^2)\,. \tag{2.2.37}$$

This complete Green function does not contain any disconnected parts because the vacuum is trivial in perturbation theory; i.e.,

$$\frac{\langle\tilde{0}|\hat{\phi}(x)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} := G(x)|_{J=0} = \frac{1}{i}\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = 0\,, \tag{2.2.38}$$

so that the field does not have a nonzero vacuum expectation value. (NB. This is the simplest demonstration of the fact that dynamical symmetry breaking is a phenomenon inaccessible in perturbation theory.)

2.3 Fermionic Quantum Field

Finitely Many Degrees of Freedom

Fermionic fields do not have a classical analogue: classical physics does not contain anticommuting fields. In order to treat fermions using functional integrals one must employ Grassmann variables. Reference [BE66] is the standard source for a rigorous discussion of Grassmann algebras. Here we will only review some necessary ideas.

The Grassmann algebra G_N is generated by the set of N elements, $\theta_1\,,\ldots\,,\theta_N$, which satisfy the anticommutation relations

$$\{\theta_i,\theta_j\} = 0\,,\ i,j = 1,2,\ldots,N\,. \tag{2.3.1}$$

It is clear from Eq. (2.3.1) that $\theta_i^2 = 0$ for $i = 1,\ldots,N$. In addition, the elements $\{\theta_i\}$ provide the source for the basis vectors of a 2^n-dimensional space, spanned by the monomials:

$$1,\ \theta_1,\ldots,\theta_N,\ \theta_1\theta_2,\ldots\theta_{N-1}\theta_N,\ldots,\ \theta_1\theta_2\ldots\theta_N\,; \tag{2.3.2}$$

i.e., G_N is a 2^N-dimensional vector space. (NB. One can always choose the p-degree monomial in Eq. (2.3.2): $\theta_{i_1}\theta_{i_2}\ldots\theta_{I_N}$, such that $i_1 < i_2 < \ldots < i_N$.) Obviously, any element $f(\theta) \in G_N$ can be written

$$\begin{aligned} f(\theta) &= f_0 + \sum_{i_1} f_1(i_1)\,\theta_{i_1} + \sum_{i_1,i_2} f_2(i_1,i_2)\,\theta_{i_1}\theta_{i_2} + \ldots \\ &\quad + \sum_{i_1,i_2,\ldots,i_N} f_N(i_1,i_2,\ldots,i_N)\,\theta_1\theta_2\ldots\theta_N\,, \end{aligned} \tag{2.3.3}$$

where the coefficients $f_p(i_1,i_2,\ldots,i_p)$ are unique if they are chosen to be fully antisymmetric for $p \geq 2$.

Both "left" and "right" derivatives can be defined on G_N. As usual, they are linear operators and hence it suffices to specify their operation on the basis elements:

$$\begin{aligned} &\frac{\partial}{\partial\theta_s}\,\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p} \\ &\quad = \delta_{si_1}\theta_{i_2}\ldots\theta_{i_p} - \delta_{si_2}\theta_{i_1}\ldots\theta_{i_p} + \ldots + (-)^{p-1}\delta_{si_p}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_{p-1}}\,, \end{aligned} \tag{2.3.4}$$

$$\begin{aligned} &\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}\frac{\overleftarrow{\partial}}{\partial\theta_s} \\ &\quad = \delta_{si_p}\theta_{i_1}\ldots\theta_{i_{p-1}} - \delta_{si_{p-1}}\theta_{i_1}\ldots\theta_{i_{p-2}}\theta_{i_p} + \ldots + (-)^{p-1}\delta_{si_1}\theta_{i_2}\theta_{i_p}\,. \end{aligned} \tag{2.3.5}$$

The operation on a general element, $f(\theta) \in G_N$, is easily obtained. It is also obvious that

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\,f(\theta) = -\,\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}f(\theta)\,. \tag{2.3.6}$$

A definition of integration requires the introduction of Grassmannian line elements: $d\theta_i$, $i = 1,\ldots,N$. These elements also satisfy Grassmann algebras:

$$\{d\theta_i, d\theta_j\} = 0 = \{\theta_i, d\theta_j\}\,,\ i,j = 1,2,\ldots N\,. \tag{2.3.7}$$

The integral calculus is completely defined by the following two identities:

$$\int d\theta_i = 0\,,\ \int d\theta_i\,\theta_i = 1\,,\ i = 1,2,\ldots,N\,. \tag{2.3.8}$$

For example, it is straightforward to prove, using Eq. (2.3.3),

$$\int d\theta_N \dots d\theta_1 \, f(\theta) = N! \, f_N(1, 2, \dots, N) \, . \tag{2.3.9}$$

In standard integral calculus a change of integration variables is often used to simplify an integral. That operation can also be defined in the present context. Consider a nonsingular matrix (K_{ij}), $i, j = 1, \dots, N$, and define new Grassmann variables $\xi_1, \dots, \xi_N$ via

$$\theta_i = \sum_{i=j}^{N} K_{ij} \, \xi_j \, . \tag{2.3.10}$$

With the definition

$$d\theta_i = \sum_{j=1}^{N} (K^{-1})_{ji} \, d\xi_j \tag{2.3.11}$$

one guarantees

$$\int d\theta_i \, \theta_j = \delta_{ij} = \int d\xi_i \, \xi_j \, . \tag{2.3.12}$$

It follows immediately that

$$\theta_1 \theta_2 \dots \theta_N = (\det K) \, \xi_1 \xi_2 \dots \xi_N \tag{2.3.13}$$

$$d\theta_N d\theta_{N-1} \dots d\theta_1 = (\det K^{-1}) \, d\xi_N d\xi_{N-1} \dots d\xi_1 \, , \tag{2.3.14}$$

and hence

$$\int d\theta_N \dots d\theta_1 \, f(\theta) = (\det K^{-1}) \int d\xi_N \dots d\xi_1 \, f(\theta(\xi)) \, . \tag{2.3.15}$$

In analogy with scalar field theory, for fermions one expects to encounter integrals of the type

$$I := \int d\theta_N \dots d\theta_1 \, \exp \left\{ \sum_{i,j=1}^{N} \theta_i A_{ij} \theta_j \right\} \, , \tag{2.3.16}$$

where (A_{ij}) is an antisymmetric matrix. NB. Any symmetric part of the matrix, A, cannot contribute since:

$$\sum_{i,j} \theta_i A_{ij} \theta_j \quad \stackrel{\text{relabel}}{=} \quad \sum_{j,i} \theta_j A_{ji} \theta_i \tag{2.3.17}$$

$$\overset{\text{use sym.}}{=} \sum_{i,j} \theta_j A_{ij} \theta_i \tag{2.3.18}$$

$$\overset{\text{anticom.}}{=} -\sum_{i,j} \theta_i A_{ij} \theta_j \,. \tag{2.3.19}$$

$$\therefore \sum_{i,j} \theta_i A_{ij} \theta_j = 0 \quad \text{for } A = A^{\mathrm{t}} \,. \tag{2.3.20}$$

Assume for the moment that A is a real matrix. Then there is an orthogonal matrix S ($SS^{\mathrm{t}} = I$) for which

$$S^{\mathrm{t}} A S = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \dots \\ -\lambda_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_2 & \dots \\ 0 & 0 & -\lambda_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} =: \tilde{A} \,. \tag{2.3.21}$$

Consequently, applying the linear transformation $\theta_i = \sum_{i=1}^{N} S_{ij}\, \xi_j$ and using Eq. (2.3.15), we obtain

$$I = \int d\xi_N \dots d\xi_1 \, \exp\left\{ \sum_{i,j=1}^{N} \xi_i \tilde{A}_{ij} \xi_j \right\} . \tag{2.3.22}$$

Hence

$$I = \begin{cases} \int d\xi_N \dots d\xi_1 \, \exp\left\{2\left[\lambda_1\xi_1\xi_2 + \lambda_2\xi_3\xi_4 + \dots + \lambda_{N/2}\xi_{N-1}\xi_N\right]\right\} \\ \qquad = 2^{N/2}\lambda_1\lambda_2\dots\lambda_N \,, \ N \text{ even} \\ \int d\xi_N \dots d\xi_1 \, \exp\left\{2\left[\lambda_1\xi_1\xi_2 + \lambda_2\xi_3\xi_4 + \dots + \lambda_{(N-1)/2}\xi_{N-2}\xi_{N-1}\right]\right\} \\ \qquad = 0 \,, \ N \text{ odd} \end{cases} \tag{2.3.23}$$

i.e., since $\det A = \det \tilde{A}$,

$$I = \sqrt{\det 2A} \,. \tag{2.3.24}$$

Equation (2.3.24) is valid for any real matrix, A. Hence, by the analytic function theorem, it is also valid for any complex matrix A.

The Lagrangian density associated with the Dirac equation involves a field $\bar{\psi}$, which plays the role of a conjugate to ψ. If we are to express ψ as a vector in G_N then we will need a conjugate space in which $\bar{\psi}$ is defined. Hence it is necessary to define $\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_N$ such that the operation $\theta_i \leftrightarrow \bar{\theta}_i$

is an involution of the algebra onto itself with the following properties:

$$\begin{array}{ll} i) & \overline{(\bar{\theta}_i)} = \theta_i \\ ii) & \overline{(\theta_i \theta_j)} = \bar{\theta}_j \, \bar{\theta}_i \\ iii) & \overline{\lambda \, \theta_i} = \lambda^* \, \bar{\theta}_i \, , \; \lambda \in \mathbb{C} \, . \end{array} \tag{2.3.25}$$

The elements of the Grassmann algebra with involution are θ_1, θ_2, ..., θ_N, $\bar{\theta}_1$, $\bar{\theta}_2$ ..., $\bar{\theta}_N$, each anticommuting with every other. Defining integration via obvious analogy with Eq. (2.3.8), it follows that

$$\int d\bar{\theta}_N d\theta_N \, \ldots \, d\theta_1 d\theta_1 \, \exp \left\{ - \sum_{i,j=1}^{N} \bar{\theta}_i B_{ij} \theta_j \right\} = \det B \, , \tag{2.3.26}$$

for any matrix B. (NB. This is the origin of the fermion determinant in the quantum field theory of fermions.) This may be compared with the analogous result for commuting real numbers, Eq. (2.2.21):

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} dx_i \, \exp\{-\pi \sum_{i,j=1}^{N} x_i \, A_{ij} \, x_j\} = \frac{1}{\sqrt{\det A}} \, . \tag{2.3.27}$$

Quantum Field Theory

To describe a fermionic quantum field the preceding analysis must be generalised to the case of infinitely many generators. A rigorous discussion can be found in Ref. [BE66] but here we will simply motivate the extension via plausible but merely formal manipulations.

Suppose the functions $\{u_n(x) \, , \; n = 0, \ldots, \infty\}$ are a complete, orthonormal set that span a given Hilbert space and consider the Grassmann function

$$\theta(x) := \sum_{n=0}^{\infty} u_n(x) \, \theta_n \, , \tag{2.3.28}$$

where $\{\theta_n\}$ are Grassmann variables. Clearly

$$\{\theta(x), \theta(y)\} = 0 \, . \tag{2.3.29}$$

The elements $\theta(x)$ are considered to be the generators of the "Grassman algebra" G and, in complete analogy with Eq. (2.3.3), any element of G

can be written uniquely as

$$f = \sum_{n=0}^{\infty} \int dx_1 dx_2 \dots dx_N \, \theta(x_1)\theta(x_2)\dots\theta(x_N)\, f_n(x_1, x_2, \dots, x_N)\,, \tag{2.3.30}$$

where, for $N \geq 2$, the coefficients $f_n(x_1, x_2, \dots, x_N)$ are fully antisymmetric functions of their arguments.

In another analogy, the left- and right-functional-derivatives are defined via their action on the basis vectors:

$$\frac{\delta}{\delta\theta(x)}\,\theta(x_1)\theta(x_2)\dots\theta(x_n) = \\ \delta(x-x_1)\,\theta(x_2)\dots\theta(x_n) - \dots + (-)^{n-1}\delta(x-x_n)\,\theta(x_1)\dots\theta(x_{n-1})\,, \tag{2.3.31}$$

$$\theta(x_1)\theta(x_2)\dots\theta(x_n)\,\frac{\overleftarrow{\delta}}{\delta\theta(x)} = \\ \delta(x-x_n)\,\theta(x_1)\dots\theta(x_{n-1}) - \dots + (-)^{n-1}\delta(x-x_1)\,\theta(x_2)\dots\theta(x_n)\,, \tag{2.3.32}$$

cf. Eqs. (2.3.4), (2.3.5).

Finally, we can extend the definition of integration. Denoting

$$[\mathcal{D}\theta(x)] := \lim_{N\to\infty} d\theta_N \dots d\theta_2 d\theta_1\,, \tag{2.3.33}$$

consider the "standard" Gaussian integral

$$I := \int [\mathcal{D}\theta(x)]\, \exp\left\{\int dxdy\, \theta(x)A(x,y)\theta(y)\right\} \tag{2.3.34}$$

where, clearly, only the antisymmetric part of $A(x,y)$ can contribute to the result. Define

$$A_{ij} := \int dxdy\, u_i(x)A(x,y)u_j(y)\,, \tag{2.3.35}$$

then

$$I = \lim_{N\to\infty} \int d\theta_N \dots d\theta_2 d\theta_1\, \exp\left\{\sum_{i=1}^{N} \theta_i A_{ij}\theta_j\right\} \tag{2.3.36}$$

so that, using Eq. (2.3.24),

$$I = \lim_{N\to\infty} \sqrt{\det 2A_N}\,, \tag{2.3.37}$$

where, obviously, A_N is the $N \times N$ matrix in Eq. (2.3.35). This provides a definition for the formal result:

$$I = \int [\mathcal{D}\theta(x)]\, \exp\left\{ \int dxdy\, \theta(x) A(x,y) \theta(y) \right\} = \sqrt{\mathrm{Det}\, 2A}\,, \tag{2.3.38}$$

where we will subsequently identify functional equivalents of matrix operations as proper nouns; e.g., "det $\to$ Det." The result is independent of the basis vectors since all such vectors are unitarily equivalent and the determinant is cylic.*

In quantum field theory one employs a Grassmann algebra with an involution. In this case, defining the functional integral via

$$[\mathcal{D}\bar{\theta}(x)][\mathcal{D}\theta(x)] := \lim_{N\to\infty} d\bar{\theta}_N d\theta_N \ldots d\bar{\theta}_2 d\theta_2\, d\bar{\theta}_1 d\theta_1\,, \tag{2.3.39}$$

one arrives immediately at a generalisation of Eq. (2.3.26)

$$\int [\mathcal{D}\bar{\theta}(x)][\mathcal{D}\theta(x)] \exp\left\{ - \int dxdy\, \bar{\theta}(x)\, B(x,y)\, \theta(x) \right\} = \mathrm{Det} B\,, \tag{2.3.40}$$

which is also a definition.

The relation

$$\ln \det B = \mathrm{tr} \ln B\,, \tag{2.3.41}$$

which is valid for any nonsingular, finite dimensional matrix, has a generalisation that is often used in analysing quantum field theories with fermions. Its utility is to make possible a representation of the fermionic determinant as part of the quantum field theory's action via

$$\mathrm{Det} B = \exp\left\{ \mathrm{Tr}\, \mathrm{Ln} B \right\}\,. \tag{2.3.42}$$

We note that for an integral operator $O(x,y)$

$$\mathrm{Tr}\, O(x,y) := \int d^4x\, \mathrm{tr}\, O(x,x)\,, \tag{2.3.43}$$

*This means that a new basis is always related to another basis via $\boldsymbol{u'} = U\boldsymbol{u}$, with $UU^\dagger = \boldsymbol{I}$. Transforming to a new basis therefore introduces a modified exponent, now involving the matrix $UAU^\dagger$, but the result is unchanged because $\det UAU^\dagger = \det A$.

which is an obvious analogy to the definition for finite-dimensional matrices. Furthermore, a functional of an operator, whenever it is well-defined, is obtained via the function's power series; i.e., if

$$f(x) = f_0 + f_1\, x + f_2\, x^2 + [\ldots]\,, \tag{2.3.44}$$

then

$$f[O(x,y)] = f_0\, \delta^4(x-y) + f_1\, O(x,y) + f_2 \int d^4w\, O(x,w) O(w,y) + [\ldots]\,. \tag{2.3.45}$$

Generating Functional for Free Dirac Fields

The Lagragian density for the free Dirac field is

$$L_0^{\psi}(x) = \int d^4x\, \bar{\psi}(x)\, (i\partial\!\!\!/ - m)\, \psi(x)\,. \tag{2.3.46}$$

Consider therefore the functional integral

$$W[\bar{\xi}, \xi] = \int [\mathcal{D}\bar{\psi}(x)][\mathcal{D}\psi(x)]\ \mathrm{e}^{i\int d^4x \left[\bar{\psi}(x)\left(i\partial\!\!\!/ - m + i\eta^+\right)\psi(x) + \bar{\psi}(x)\xi(x) + \bar{\xi}(x)\psi(x)\right]}. \tag{2.3.47}$$

Here $\bar{\psi}(x)$, $\psi(x)$ are identified with the generators of G, with the minor additional complication that the spinor degree-of-freedom is implicit; i.e., to be explicit, one should write

$$\prod_{r=1}^{4} [\mathcal{D}\bar{\psi}_r(x)] \prod_{s=1}^{4} [\mathcal{D}\psi_s(x)]. \tag{2.3.48}$$

This only adds a finite matrix degree-of-freedom to the problem, so that "Det A" will mean both a functional *and* a matrix determinant. This effect will be encountered again; e.g., with the appearance of fermion colour and flavour. In Eq. (2.3.47) we have also introduced anticommuting sources: $\bar{\xi}(x)$, $\xi(x)$, which are also elements in the Grassmann algebra, G.

The free-field generating functional involves a Gaussian integral. To evaluate the integral we write

$$O(x,y) = (i\partial\!\!\!/ - m + i\eta^+)\delta^4(x-y) \tag{2.3.49}$$

and observe that the solution of

$$\int d^4w\, O(x,w)\, P(w,y) = \boldsymbol{I}\, \delta^4(x-y) \tag{2.3.50}$$

i.e., the inverse of the operator $O(x,y)$ is (see Eq. 1.4.5) precisely the free-fermion propagator:

$$P(x,y) = S_0(x-y)\,. \tag{2.3.51}$$

Hence we can rewrite Eq. (2.3.47) in the form

$$W[\bar\xi,\xi] = \int [\mathcal{D}\bar\psi(x)][\mathcal{D}\psi(x)]\, \mathrm{e}^{i\int d^4x d^4y \left[\bar\psi'(x)O(x,y)\psi'(y) - \bar\xi(x)S_0(x-y)\xi(y)\right]}, \tag{2.3.52}$$

where

$$\begin{array}{rcl} \bar\psi'(x) & := & \bar\psi(x) + \int d^4w\, \bar\xi(w)\, S_0(w-x)\,, \\ \psi(x) & := & \psi(x) + \int d^4w\, S_0(x-w)\, \xi(w)\,. \end{array} \tag{2.3.53}$$

Clearly, $\bar\psi'(x)$ and $\psi'(x)$ are still in G and hence related to the original variables by a unitary transformation. Thus changing to the "primed" variables introduces a unit Jacobian and so

$$\begin{array}{rcl} W[\bar\xi,\xi] & = & \mathrm{e}^{-i\int d^4x d^4y\, \bar\xi(x)\, S_0(x-y)\, \xi(y)} \\ & & \times \displaystyle\int [\mathcal{D}\bar\psi'(x)][\mathcal{D}\psi'(x)]\; \mathrm{e}^{i\int d^4x d^4y\, \bar\psi'(x)O(x,y)\psi'(y)} \\ & = & \mathrm{Det}[-iS_0^{-1}(x-y)]\; \mathrm{e}^{-i\int d^4x d^4y\, \bar\xi(x)\, S_0(x-y)\, \xi(y)} \\ & = & \dfrac{1}{\mathcal{N}_\prime^\psi}\; \mathrm{e}^{-i\int d^4x d^4y\, \bar\xi(x)\, S_0(x-y)\, \xi(y)}, \end{array} \tag{2.3.54}$$

where $\mathcal{N}_0^\psi := \mathrm{Det}[iS_0(x-y)]$. Clearly.

$$\mathcal{N}_0^\psi \;\; W[\bar\xi,\xi]\big|_{\bar\xi=0=\xi} = 1\,. \tag{2.3.55}$$

The 2 point Green function for the free-fermion quantum field theory is now easily obtained:

$$\begin{array}{rcl} \mathcal{N}_0^\psi \;\; \dfrac{\delta^2 W[\bar\xi,\xi]}{i\delta\bar\xi(x)\,(-i)\delta\xi(y)}\Big|_{\bar\xi=0=\xi} & = & \dfrac{\langle 0|T\{\hat\psi(x)\hat{\bar\psi}(y)\}|0\rangle}{\langle 0|0\rangle} \\ & = & \displaystyle\int [\mathcal{D}\bar\psi(x)][\mathcal{D}\psi(x)]\; \psi(x)\bar\psi(y)\, \mathrm{e}^{i\int d^4x\, \bar\psi(x)\left(i\partial\!\!\!/ - m + i\eta^+\right)\psi(x)} \\ & = & i\, S_0(x-y)\,; \end{array} \tag{2.3.56}$$

i.e., the inverse of the Dirac operator, with exactly the Feynman boundary conditions.

As in the example of a scalar quantum field theory, the generating functional for *connected* n-point Green functions is $Z[\bar{\xi},\xi]$, defined via:

$$W[\bar{\xi},\xi] =: \exp\left\{iZ[\bar{\xi},\xi]\right\} . \tag{2.3.57}$$

Hitherto we have not illustrated what is meant by "Det$O$," where O is an integral operator. We will now provide a formal example. (Rigour requires a careful consideration of regularisation and limits.) Consider a translationally invariant operator

$$O(x,y) = O(x-y) = \int \frac{d^4p}{(2\pi)^4}\, O(p)\, \mathrm{e}^{-i(p,x-y)}. \tag{2.3.58}$$

Then, for f as in Eq. (2.3.45),

$$\begin{aligned} f[O(x-y)] &= \int \frac{d^4p}{(2\pi)^4}\, \left\{f_0 + f_1\, O(p) + f_2\, O^2(p) + [\ldots]\right\}\, \mathrm{e}^{-i(p,x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}\, f(O(p))\, \mathrm{e}^{-i(p,x-y)} . \end{aligned} \tag{2.3.59}$$

We now apply this to $\mathcal{N}_0^{\psi} := \mathrm{Det}[iS_0(x-y)]$ and observe that Eq. (2.3.42) means one can begin by considering $\mathrm{TrLn}\, iS_0(x-y)$. Writing

$$S_0(p) = m\, \Delta_0(p^2) \left[1 + \frac{\not{p}}{m}\right] ,\ \Delta_0(p^2) = \frac{1}{p^2 - m^2 + i\eta^+} , \tag{2.3.60}$$

the free fermion propagator can be re-expressed as a product of integral operators:

$$S_0(x-y) = \int d^4w\, m\, \Delta_0(x-w)\, \mathcal{F}(w-y) , \tag{2.3.61}$$

with $\Delta_0(x-y)$ given in Eq. (2.2.27) and

$$\mathcal{F}(x-y) = \int \frac{d^4p}{(2\pi)^4} \left[1 + \frac{\not{p}}{m}\right] \mathrm{e}^{-i(p,x-y)} . \tag{2.3.62}$$

It follows that

$$\mathrm{TrLn}\, iS_0(x-y) = \mathrm{Tr}\left\{\mathrm{Ln}\, i\, m\Delta_0(x-y) + \mathrm{Ln}\left[\delta^4(x-y) + \mathcal{F}(x-y)\right]\right\} . \tag{2.3.63}$$

Using Eqs. (2.3.43), (2.3.59), one can express the second term as

$$\mathrm{TrLn}\left[\delta^4(x-y) + \mathcal{F}(x-y)\right]$$

$$
\begin{aligned}
&= \int d^4x \int \frac{d^4p}{(2\pi)^4} \,\mathrm{tr}\ln\left[1+\mathcal{F}(p)\right] \\
&= \int d^4x \int \frac{d^4p}{(2\pi)^4}\, 2\,\ln\left[1-\frac{p^2}{m^2}\right] \qquad (2.3.64)
\end{aligned}
$$

and, applying the same equations, the first term is

$$
\mathrm{TrLn}\, im\Delta_0(x-y) = \int d^4x \int \frac{d^4p}{(2\pi)^4}\, 2\,\ln\left[i\,m\Delta_0(p^2)\right]^2 , \qquad (2.3.65)
$$

where in both cases $\int d^4x$ measures the (infinite) spacetime volume. Combining these results one obtains

$$
\mathrm{Ln}\,\mathcal{N}_0^{\psi} = \mathrm{TrLn}\, iS_0(x-y) = \int d^4x \int \frac{d^4p}{(2\pi)^4}\, 2\,\ln\Delta_0(p^2)\,, \qquad (2.3.66)
$$

where the factor of 2 reflects the spin-degeneracy of the free-fermion's eigenvalues. (Including a "colour" degree-of-freedom, this would become "$2N_c$," where N_c is the number of colours.)

2.4 Gauge Fields

To begin we will consider a non-Abelian gauge field theory in the absence of couplings to matter fields, which is described by a Lagrangian density:

$$
\begin{aligned}
L(x) &= -\tfrac{1}{4}F^a_{\mu\nu}(x)F_a^{\mu\nu}(x)\,, \qquad (2.4.1)\\
F^a_{\mu\nu}(x) &= \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g f_{abc} B^b_\mu(x) B^c_\nu(x)\,, \qquad (2.4.2)
\end{aligned}
$$

where g is a coupling constant and f_{abc} are the structure constants of $SU(N_c)$; i.e., with $\{T^a\,; a = 1,\ldots,N_c^2-1\}$ denoting the generators of the group

$$
[T_a, T_b] = i f_{abc} T_c\,. \qquad (2.4.3)
$$

In the fundamental representation $\{T_a\} := \{\frac{\lambda^a}{2}\}$, where $\{\lambda^a\}$ are the generalisations of the eight Gell-Mann matrices, while in the adjoint representation, relevant to the realisation of transformations on the gauge fields,

$$
(T^a)_{bc} = -i f_{abc}\,. \qquad (2.4.4)
$$

An obvious guess for the form of the generating functional is

$$W[J] = \int [\mathcal{D}B^a_\mu]\, \mathrm{e}^{i\int d^4x\, \left[L(x)+J^\mu_a(x)\, B^a_\mu(x)\right]}\,, \tag{2.4.5}$$

where, as usual, $J^\mu_a(x)$ is a (classical) external source for the gauge field. It will immediately be observed that this is a Lagrangian-based expression for the generating functional, even though the Hamiltonian derived from Eq. (2.4.1) is not of the form in Eq. (2.2.13). It is nevertheless (almost) correct (we will motivate the modifications that need to be made to make it completely correct) and provides the foundation for a manifestly Poincaré covariant quantisation of the field theory. Alternatively, one could work with Coulomb gauge, build the Hamiltonian and construct $W[J]$ in the canonical fashion, as described in Sec. 2.2, but then covariance is lost. The Coulomb gauge procedure gives the same S-matrix elements (Green functions) as the (corrected-) Lagrangian formalism and hence they are completely equivalent. However, manifest covariance is extremely useful as it often simplifies the allowed form of Green functions and certainly simplifies the calculation of cross-sections. Hence the Lagrangian formulation is most often used.

The primary fault with Eq. (2.4.5) is that the free-field part of the Lagrangian density is singular; i.e., the determinant encountered in evaluating the free-gauge-boson generating functional vanishes, and hence the operator cannot be inverted.

This is easily demonstrated. Observe that

$$\begin{aligned}
\int d^4x\, L_0(x) &= -\frac{1}{4}\int d^4x\, [\partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x)][\partial^\mu B^\nu_a(x) - \partial^\nu B^\mu_a(x)] \\
&= -\frac{1}{2}\int d^4x\, d^4y\, B^a_\mu(x)\left\{[-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu]\delta^4(x-y)\right\} B^a_\nu(y) \\
&=: \frac{1}{2}\int d^4x\, d^4y\, B^a_\mu(x)\, K^{\mu\nu}(x,y)\, B^a_\nu(y)\,.
\end{aligned} \tag{2.4.6}$$

The operator $K^{\mu\nu}(x,y)$ thus determined can be expressed

$$K^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4}\, \left(-g^{\mu\nu}q^2 + q^\mu q^\nu\right) \mathrm{e}^{-i(p,x-y)} \tag{2.4.7}$$

from which it is apparent that the Fourier amplitude is a *projection* operator; i.e., a key feature of $K^{\mu\nu}(x-y)$ is that it projects onto the space of

transverse gauge field configurations:

$$q_\mu \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right) = 0 = \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right) q_\nu \,, \tag{2.4.8}$$

where q^μ is the four-momentum associated with the gauge field. It follows that the $W_0[J]$ obtained from Eq. (2.4.5) has no damping associated with longitudinal gauge fields and is therefore meaningless. A simple analogy is

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \mathrm{e}^{-(x-y)^2} \,, \tag{2.4.9}$$

in which the integrand does not damp along trajectories in the $\{x, y\}$-plane related via a spatial translation: $\{x, y\} \to \{x + a, y + a\}$. Hence there is an overall divergence associated with the translation of the centre of momentum: $X = (x + y)/2$, $Y = (x - y)$; $X \to X + \{a, a\}$, $Y \to Y$,

$$\begin{aligned} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \mathrm{e}^{-(x-y)^2} &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY \, \mathrm{e}^{-Y^2} \\ &= \left(\int_{-\infty}^{\infty} dX \right) \sqrt{\pi} = \infty \,. \end{aligned} \tag{2.4.10}$$

The underlying problem, which is signalled by the behaviour just identified, is the *gauge invariance* of the action: $\int d^4x \, L(x)$; i.e., the action is invariant under a local field transformation

$$B_\mu(x) := ig \, B^a_\mu(x) \, T^a \to G(x) \, B_\mu(x) \, G^{-1}(x) + [\partial_\mu G(x)] \, G^{-1}(x) \,, \tag{2.4.11}$$

$$G(x) = \exp \{ -ig \, T^a \, \Theta^a(x) \} \,. \tag{2.4.12}$$

This means that, given a reference field configuration $\hat{B}_\mu(x)$, the integrand in the generating functional, Eq. (2.4.5), is constant along paths through the gauge field manifold traversed by applying the gauge transformations to $\hat{B}_\mu(x)$. Since the parameters characterising the gauge transformations, Θ^a, are continuous functions, each such gauge orbit contains an uncountable infinity of gauge field configurations. It is therefore immediately apparent that the generating functional, as written, is undefined: it contains a multiplicative factor proportional to the length (or volume) of the gauge orbit. (NB. While there is, in addition, an uncountable infinity of distinct reference configurations, the action changes upon any shift orthogonal to a gauge orbit.)

Returning to the example in Eq. (2.4.10), the analogy is that the "Lagrangian density," $\ell(x,y) = (x-y)^2$ is invariant under translations; i.e., the integrand is invariant under the operation

$$g_s(x,y) = \exp\left(s\frac{\partial}{\partial x} + s\frac{\partial}{\partial y}\right) \tag{2.4.13}$$

$$\{x,y\} \to \{x',y'\} = g_s(x,y)\,\{x,y\} = \{x+s, y+s\}\,. \tag{2.4.14}$$

Hence, given a reference point $P = \{x_0, y_0\} = \{1,0\}$, the integrand is constant along the path $\{x,y\} = P_0 + \{s,s\}$ through the $\{x,y\}$-plane. (This is a translation of the centre-of-mass: $X_0 = \{x_0+y_0\}/2 = 1/2 \to X_0 + s$.) Since $s \in (-\infty,\infty)$, this path contains an uncountable infinity of points, and at each one the integrand has precisely the same value. The integral thus contains a multiplicative factor proportional to the length of the translation path, which clearly produces an infinite (meaningless) result for the integral. (NB. The value of the integrand changes upon a translation orthogonal to that just identified.)

2.4.1 *Faddeev-Popov Determinant and Ghosts*

This problem with the functional integral over gauge fields was identified by Faddeev and Popov [FP67]. They proposed to solve the problem by identifying and extracting the gauge orbit volume factor.

Simple Model

Before describing their procedure in quantum field theory we illustrate it using the simple integral model. We begin by defining a functional of our "field variable," $\{x,y\}$, which intersects the centre-of-mass translation path once, and only once:

$$f(x,y) = (x+y)/2 - a = 0\,. \tag{2.4.15}$$

Now define a functional Δ_f such that

$$\Delta_f[x,y]\int_{-\infty}^{\infty} da\,\delta((x+y)/2-a) = 1\,. \tag{2.4.16}$$

It is clear that $\Delta_f[x,y]$ is independent of a:

$$\Delta_f[x+a', y+a']^{-1} = \int_{-\infty}^{\infty} da\,\delta((x+a'+y+a')/2-a)$$

$$\stackrel{\tilde{a}=\underline{a}-a'}{=} \int_{-\infty}^{\infty} d\tilde{a}\,\delta((x+y)/2-\tilde{a}) \\ = \Delta_f[x,y]^{-1}\,. \tag{2.4.17}$$

The model generating functional can be rewritten using Δ_f:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)} \\ = \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)}\, \Delta_f[x,y]\, \delta((x+y)/2-a) \tag{2.4.18}$$

and now one performs a centre-of-mass translation: $x \to x' = x + a$, $y \to y' = y + a$, under which the action is invariant and the integral becomes

$$\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)}\, \Delta_f[x,y]\, \delta((x+y)/2-a) \\ = \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)}\, \Delta_f[x+a,y+a]\, \delta((x+y)/2)\,. \tag{2.4.19}$$

Now making use of the a-independence of $\Delta_f[x,y]$, Eq. (2.4.17),

$$= \left[\int_{-\infty}^{\infty} da\right] \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)}\, \Delta_f[x,y]\, \delta((x+y)/2)\,. \tag{2.4.20}$$

In this last line the "volume" or "path length" has explicitly been factored out at the cost of introducing a δ-function, which fixes the centre-of-mass; i.e., a single point on the path of translationally equivalent configurations, and a functional Δ_f, which, we will see, is the analogue in this simple model of the Faddeev-Popov determinant. Hence the "correct" definition of the generating functional for this model is

$$w(\vec{j}) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\ell(x,y)+j_x x+j_y y}\, \Delta_f[x,y]\, \delta((x+y)/2)\,. \tag{2.4.21}$$

Gauge Fixing Conditions

To implement this idea for the real case of non-Abelian gauge fields one envisages an hypersurface, lying in the manifold of all gauge fields, which intersects each gauge orbit once, and only once. This means that if

$$f_a[B^a_\mu(x);x] = 0\,,\ a = 1,2,\ldots,N_c^2-1\,, \tag{2.4.22}$$

is the equation describing the hypersurface, then there is a unique element in each gauge orbit that satisfies one Eq. (2.4.22), and the set of these unique elements, none of which can be obtained from another by a gauge transformation, forms a representative class that alone truly characterises the physical configuration of gauge fields. The gauge-equivalent, and therefore redundant, elements are absent.

Equations (2.4.22) can also be viewed as defining a set of non-linear equations for $G(x)$, Eq (2.4.12). This in the sense that for a given field configuration, $B_\mu(x)$, it is always possible to find a unique gauge transformation, $G_1(x)$, that yields a gauge transformed field $B_\mu^{G_1}(x)$, from $B_\mu(x)$ via Eq. (2.4.11), which is the one and only solution of $f_a[B_\mu^{a,G_1}(x);x]=0$. Equation (2.4.22) therefore defines a gauge fixing condition.

In order for Eqs. (2.4.22) to be useful it must be possible that, when given a configuration $B_\mu(x)$ for which $f_a[B_\mu(x);x]\neq 0$, the equation

$$f_a[B_\mu^G(x);x]=0 \tag{2.4.23}$$

can be solved for the gauge transformation $G(x)$. To see a consequence of this requirement, consider a gauge configuration $B_\mu^b(x)$ that almost, but not quite, satisfies Eqs. (2.4.22). Applying an infinitesimal gauge transformation to this $B_\mu^b(x)$ the requirement entails that it must be possible to solve

$$f_a[B_\mu^b(x)\to B_\mu^b(x)-g\,f_{bdc}\,B_\mu^d(x)\,\delta\Theta^c(x)-\partial_\mu\delta\Theta^b(x);x]=0 \tag{2.4.24}$$

for the infinitesimal gauge transformation parameters $\delta\Theta^a(x)$. Equation (2.4.24) can be written (using the chain rule)

$$f_a[B_\mu^b(x)]-\int d^4y\,\frac{\delta f_a[B_\mu^b(x);x]}{\delta B_\nu^c(y)}\,\left[\delta^{cd}\partial_\nu+gf_{ced}B_\nu^e(y)\right]\delta\Theta^d(y)=0\,. \tag{2.4.25}$$

This looks like the matrix equation $\vec{f}=\mathcal{O}\,\vec{\theta}$, which has a solution for $\vec{\theta}$ if, and only if, $\det\mathcal{O}\neq 0$, and a similar constraint follows from Eq. (2.4.25): the gauge fixing conditions can be solved if, and only if,

$$\begin{aligned}\mathrm{Det}\,\mathcal{M}_f &:= \mathrm{Det}\left\{-\frac{\delta f_a[B_\mu^b(x);x]}{\delta B_\nu^c(y)}\,\left[\delta^{cd}\partial_\nu-gf_{cde}B_\nu^e(y)\right]\right\}\\ &=: \mathrm{Det}\left\{-\frac{\delta f_a[B_\mu^b(x);x]}{\delta B_\nu^c(y)}\,[D_\nu(y)]_{cd}\right\}\neq 0\,.\end{aligned} \tag{2.4.26}$$

Equation (2.4.26) is the so-called *admissibility condition* for gauge fixing constraints.

A simple illustration is provided by the lightlike (Hamilton) gauges, which are specified by

$$n^\mu B^a_\mu(x) = 0\,,\ n^2 > 1\,,\ a = 1, 2, \ldots, (N^2-1)\,. \tag{2.4.27}$$

Choosing $(n^\mu) = (1, 0, 0, 0)$, using Eq. (2.4.11), the equation for $G(x)$ is

$$\frac{\partial}{\partial t} G(t, \vec{x}) = -\, G(t, \vec{x})\, B^0(t, \vec{x})\,, \tag{2.4.28}$$

and this nonperturbative equation has the unique solution

$$G(t, \vec{x}) = T\, \exp\left\{ -\int_{-\infty}^{t} ds\, B^0(s, \vec{x}) \right\}\,, \tag{2.4.29}$$

where T is the time ordering operator. One may compare this with Eq. (2.4.25), which only provides a perturbative (in g) solution. While the existence of a complete, nonperturbative solution may be an advantage, Eq. (2.4.29) is not a Poincaré covariant constraint and that makes it difficult to employ in explicit calculations.

A number of other commonly used gauge fixing conditions are

$$\begin{array}{ll} \partial^\mu B^a_\mu(x) = 0\,, & \text{Lorentz gauge} \\ \partial^\mu B^a_\mu(x) = A^a(x)\,, & \text{Generalised Lorentz gauge} \\ n^\mu B^a_\mu(x) = 0\,,\ n^2 < 0\,, & \text{Axial gauge} \\ n^\mu B^a_\mu(x) = 0\,,\ n^2 = 0\,, & \text{Light-like gauge} \\ \vec{\nabla} \cdot \vec{B}^a(x) = 0\,, & \text{Coulomb gauge} \end{array} \tag{2.4.30}$$

and the generalised axial, light-like and Coulomb gauges, where an arbitrary function, $A^a(x)$, features on the r.h.s.

All of these choices satisfy the admissability condition, Eq. (2.4.26), for small gauge field variations but in some cases, such as Lorentz gauge, the uniqueness condition fails for large variations; i.e., those that are outside the domain of perturbation theory. This means that there are at least two solutions: G_1, G_2, of Eq. (2.4.23), and perhaps uncountably many more. Since no nonperturbative solution of any gauge field theory in four spacetime dimensions exists, the actual number of solutions is unknown. If the number is infinite then the Faddeev-Popov definition of the generating functional fails in that gauge. These additional solutions are called *Gribov Copies* and their existence raises questions about the correct way to furnish

a nonperturbative definition of the generating functional [GR78], which are currently unanswered.

Isolating the Gauge Orbit Volume

To proceed one needs a little information about the representation of non-Abelian groups. Suppose u is an element of the group $SU(N)$. Every such element can be characterised by (N^2-1) real parameters: $\{\Theta^a,\, a = 1,\ldots,N^2-1\}$. Let $G(u)$ be the representation of u under which the gauge fields transform; i.e., the adjoint representation, Eqs. (2.4.11), (2.4.12). For infinitesimal tranformations

$$G(u) = \boldsymbol{I} - igT^a\Theta^a(x) + \mathrm{O}(\Theta^2)\,, \tag{2.4.31}$$

where $\{T^a\}$ are the adjoint representation of the Lie algebra of $SU(N)$, Eq. (2.4.4). Clearly, if u, $u\prime \in SU(N)$ then $uu' \in SU(n)$ and $G(u)G(u') = G(uu')$. (These are basic properties of groups.)

To define the integral over gauge fields we must properly define the gauge-field "line element". This is the Hurwitz measure on the group space, which is invariant in the sense that $du' = d(u'u)$. In the neighbourhood of the identity one may always choose

$$du = \prod_a d\Theta^a \tag{2.4.32}$$

and the invariance means that since the integration represents a sum over all possible values of the parameters Θ^a, relabelling them as $\tilde{\Theta}^a$ cannot matter. It is now possible to quantise the gauge field; i.e., properly define Eq. (2.4.5).

Consider $\Delta_f[B^a_\mu]$ defined via

$$\Delta_f[B^b_\mu]\int\prod_x du(x)\prod_{x,a}\delta[f_a\{B^{b,u}_\mu(x);x\}] = 1\,, \tag{2.4.33}$$

cf. Eq. (2.4.16), where $B^{b,u}_\mu(x)$ is given by Eq. (2.4.11) with $G(x) \to u(x)$. $\Delta_f[B^a_\mu]$ is gauge invariant:

$$\begin{aligned}\Delta_f^{-1}[B^{b,u}_\mu] &= \int\prod_x du'(x)\prod_{x,a}\delta[f_a\{B^{b,u'u}_\mu(x);x\}]\\ &= \int\prod_x d(u'(x)u(x))\prod_{x,a}\delta[f_a\{B^{b,u'u}_\mu(x);x\}]\end{aligned}$$

$$= \int \prod_x du''(x) \prod_{x,a} \delta[f_a\{B_\mu^{b,u''}(x);x\}] = \Delta_f^{-1}[B_\mu^b]\,. \tag{2.4.34}$$

Returning to Eq. (2.4.5), one can write

$$W[0] = \int \prod_x du(x) \int [\mathcal{D}B_\mu^a]\, \Delta_f[B_\mu^a] \prod_{x,b} \delta[f_b\{B_\mu^{a,u}(x);x\}]\, \exp\{i\int d^4x\, L(x)\}\,. \tag{2.4.35}$$

Now execute a gauge transformation: $B_\mu^a(x) \to B_\mu^{a,u^{-1}}(x)$, so that, using the invariance of the measure and the action, one has

$$W[0] = \int \prod_x du(x) \int [\mathcal{D}B_\mu^a]\, \Delta_f[B_\mu^a] \prod_{x,b} \delta[f_b\{B_\mu^a(x);x\}]\, \exp\{i\int d^4x\, L(x)\}\,. \tag{2.4.36}$$

The integrand no longer depends on the group element, $u(x)$ (cf. Eq. (2.4.20)). The gauge orbit volume has thus been identified and can be eliminated so that one may define

$$W[J_\mu^a] = \int [\mathcal{D}B_\mu^a]\, \Delta_f[B_\mu^a] \prod_{x,b} \delta[f_b\{B_\mu^a(x);x\}]\, \mathrm{e}^{\,i\int d^4x\,[L(x)+J_a^\mu(x)B_\mu^a(x)]}\,. \tag{2.4.37}$$

Neglecting for a moment the possible existence of Gribov copies, Eq. (2.4.37) is the foundation we sought for a manifestly Poincaré covariant quantisation of the gauge field. However, a little more work is needed to mould a practical tool.

Ghost Fields

A first step is an explicit calculation of $\Delta_f[B_\mu^a]$. Since it always appears multiplied by a δ-function it is sufficient to evaluate $\Delta_f[B_\mu^a]$ for those field configurations that satisfy Eq. (2.4.22). Recalling Eqs. (2.4.25), (2.4.26) then for infinitesimal gauge transformations

$$\begin{aligned} f_a[B_\mu^{b,u}(x);x] &= f_a[B_\mu^b(x);x] + \int d^4y\, [\mathcal{M}_f(x,y)]_{ac}\Theta_c(y) \\ &= 0 + \int d^4y\, [\mathcal{M}_f(x,y)]_{ac}\Theta_c(y)\,. \end{aligned} \tag{2.4.38}$$

Substituting this into the definition, Eq. (2.4.33), we find

$$\Delta_f^{-1}[B_\mu^b] \;=\; \int\prod_{x,a} d\Theta^a(x)\,\delta\left[\int d^4y\,[\mathcal{M}_f(x,y)]_{ac}\Theta_c(y)\right]. \quad (2.4.39)$$

Changing variables: $\Theta^a \to \tilde{\Theta}^a = [\mathcal{M}_f]_{ac}\Theta_c$, with the integration expressing the functional product implicit, this gives

$$\Delta_f^{-1}[B_\mu^b] \;=\; (\mathrm{Det}\,\mathcal{M}_f)^{-1}\int\prod_{x,a} d\tilde{\Theta}^a(x)\,\delta\left[\tilde{\Theta}_a(x)\right], \quad (2.4.40)$$

where the determinant factor is the Jacobian of the transformation. Plainly, the integral on the r.h.s. evaluates to one (what else can the δ-functional mean?) and hence

$$\Delta_f[B_\mu^b] = \mathrm{Det}\,\mathcal{M}_f\,. \quad (2.4.41)$$

Now recall Eq. (2.3.40). This means Eq. (2.4.41) can be expressed as a functional integral over Grassmann fields: $\bar{\phi}_a$, ϕ_b, $a,b = 1,\ldots,(N^2-1)$,

$$\Delta_f[B_\mu^b] = \int[\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_a]\,\mathrm{e}^{-\int d^4x d^4y\,\bar{\phi}_b(x)\,[\mathcal{M}_f(x,y)]_{ba}\,\phi_a(y)}. \quad (2.4.42)$$

Consequently, absorbing non-dynamical constants into the normalisation,

$$W[J_\mu^a] = \int[\mathcal{D}B_\mu^a]\,[\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_a]\prod_{x,b}\delta[f_b\{B_\mu^a(x);x\}]$$
$$\times\,\mathrm{e}^{\,i\int d^4x\,[L(x)+J_a^\mu(x)B_\mu^a(x)]+\,i\int d^4xd^4y\,\bar{\phi}_b(x)\,[\mathcal{M}_f(x,y)]_{ba}\,\phi_a(y)}. \quad (2.4.43)$$

The Grassmann fields $\{\bar{\phi}_a,\phi_b\}$ are the Faddeev-Popov *Ghosts*. They are an essential consequence of gauge fixing.

As one concrete example, consider the Lorentz gauge, Eq. (2.4.30), for which

$$(\mathcal{M}_\mathrm{L})_{ab} = \delta^4(x-y)\left[\delta^{ab}\,\partial^2 - g f_{abc}\,\partial^\mu\,B_\mu^c(x)\right] \quad (2.4.44)$$

and therefore

$$W[J_\mu^a] = \int[\mathcal{D}B_\mu^a]\,[\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_a]\prod_{x,a}\delta[\partial^\mu B_\mu^a(x)]$$
$$\times\exp\left\{i\int d^4x\,\left[J_a^\mu(x)B_\mu^a(x) - \tfrac{1}{4}F_a^{\mu\nu}(x)\,F_{\mu\nu}^a(x)\right.\right.$$

$$- \partial_\mu \bar{\phi}_a(x)\, \partial^\nu \phi_a(x) + g f_{abc} [\partial^\mu \bar{\phi}_a(x)] \phi_b(x) B^c_\mu(x) \Big] \Big\} \,. \quad (2.4.45)$$

This expression makes clear that a general consequence of the Faddeev-Popov procedure is to introduce a coupling between the gauge field and the ghosts. Thus the ghosts, and hence gauge fixing, can have a direct impact on the behaviour of gauge field Green functions.

As another, consider the axial gauge, for which

$$(\mathcal{M}_A)_{ab} = \delta^4(x-y) \left[\delta^{ab}\, n_\mu\, \partial^\mu \right] . \quad (2.4.46)$$

Expressing the related determinant via ghost fields it is immediately apparent that with this choice there is no coupling between the ghosts and the gauge field quanta. Hence the ghosts decouple from the theory and may be discarded because they play no dynamical role. However, there is a cost, as always: in this gauge the effect of the delta-function, $\prod_{x,a} \delta[n^\mu B^a_\mu(x)]$, in the functional integral is to complicate the Green functions by making them depend explicitly on (n^μ). Even the free-field 2-point function exhibits such a dependence.

It is important to note that this decoupling of the ghost fields is tied to an "accidental" elimination of the f_{abc} term in $D_\mu(y)$, Eq. (2.4.26). That term is always absent in Abelian gauge theories, for which quantum electrodynamics is the archetype, because all generators commute and analogues of f_{abc} must vanish. Hence in Abelian gauge theories ghosts decouple in every gauge.

To see how $\delta[f_b\{B^a_\mu(x); x\}]$ influences the form of Green functions, consider the generalised Lorentz gauge:

$$\partial^\mu\, B^a_\mu(x) = A^a(x)\,, \quad (2.4.47)$$

where $\{A^a(x)\}$ are arbitrary functions. The Faddeev-Popov determinant is the same in generalised Lorentz gauges as it is in Lorentz gauge and hence

$$\Delta_{\rm GL}[B^a_\mu] = \Delta_{\rm L}[B^a_\mu]\,, \quad (2.4.48)$$

where the r.h.s. is given in Eq. (2.4.44). The generating functional in this gauge is therefore

$$W[J^a_\mu] = \int [\mathcal{D}B^a_\mu]\, [\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_a] \prod_{x,a} \delta[\partial^\mu B^a_\mu(x) - A^a(x)]$$

$$\times \exp\left\{ i \int d^4x \, \left[J^\mu_a(x) B^a_\mu(x) - \frac{1}{4} F^{\mu\nu}_a(x) \, F^a_{\mu\nu}(x) \right.\right.$$
$$\left.\left. - \partial_\mu \bar{\phi}_a(x) \, \partial^\nu \phi_a(x) + g f_{abc} [\partial^\mu \bar{\phi}_a(x)] \phi_b(x) B^c_\mu(x) \right] \right\} .$$

Gauge invariance of the generating functional, Eq. (2.4.33), means that one can integrate over the $\{A^a(x)\}$, with a weight function to ensure convergence:

$$\int [\mathcal{D}A^a] \, \mathrm{e}^{-\frac{i}{2\lambda} \int d^4x \, A^a(x) \, A^a(x)} , \tag{2.4.49}$$

to arrive finally at the generating functional in a covariant Lorentz gauge, specified by the parameter λ:

$$W[J^a_\mu, \bar{\xi}^a_g, \xi^a_g] = \int [\mathcal{D}B^a_\mu] \, [\mathcal{D}\bar{\phi}_b] [\mathcal{D}\phi_a]$$
$$\times \exp\left\{ i \int d^4x \, \left[-\frac{1}{4} F^{\mu\nu}_a(x) \, F^a_{\mu\nu}(x) - \frac{1}{2\lambda} [\partial^\mu B^a_\mu(x)] \, [\partial^\nu B^a_\nu(x)] \right.\right.$$
$$- \partial_\mu \bar{\phi}_a(x) \, \partial^\nu \phi_a(x) + g f_{abc} [\partial^\mu \bar{\phi}_a(x)] \phi_b(x) B^c_\mu(x)$$
$$\left.\left. + J^\mu_a(x) B^a_\mu(x) + \bar{\xi}^a_g(x) \phi^a(x) + \bar{\phi}^a(x) \xi^a_g(x) \right] \right\} , \tag{2.4.50}$$

where $\{\bar{\xi}^a_g, \xi^a_g\}$ are anticommuting external sources for the ghost fields. (NB. To complete the definition one should add convergence terms, $i\eta^+$, for every field or, preferably, work in Euclidean space.)

Observe that the free gauge boson piece of the action in Eq. (2.4.50) is

$$\frac{1}{2} \int d^4x \, d^4y \, B^a_\mu(x) \, K^{\mu\nu}(x-y;\lambda) \, B^a_\nu(y)$$
$$:= \frac{1}{2} \int d^4x \, d^4y \, B^a_\mu(x) \left\{ [g^{\mu\nu} \partial^2 - (1 - \frac{1}{\lambda}) \partial^\mu \partial^\nu - g^{\mu\nu} \, i\eta^+] \delta^4(x-y) \right\} B^a_\nu(y). \tag{2.4.51}$$

The operator $K^{\mu\nu}(x-y;\lambda)$ thus defined can be expressed

$$K^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} \left(-g^{\mu\nu}(q^2 + i\eta^+) + q^\mu q^\nu [1 - \frac{1}{\lambda}] \right) \mathrm{e}^{-i(q, x-y)}, \tag{2.4.52}$$

cf. Eq. (2.4.7), and now

$$q_\mu \, K^{\mu\nu} = - \frac{1}{\lambda} \, q^\nu \tag{2.4.53}$$

so that in this case the action does damp variations in the longitudinal components of the gauge field. $K^{\mu\nu}(x-y;\lambda)$ is the inverse of the free gauge boson propagator; i.e., the free gauge boson 2-point Green function, $D^{\mu\nu}(x-y)$, is obtained via

$$\int d^4w\, K^{\mu}_{\rho}(x-w)\, D^{\rho\nu}(w-y) = g^{\mu\nu}\,\delta^4(x-y)\,, \tag{2.4.54}$$

and hence

$$D^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4}\,\left(-g^{\mu\nu} + (1-\lambda)\frac{q^\mu q^\nu}{q^2+i\eta^+}\right)\frac{1}{q^2+i\eta^+}\,\mathrm{e}^{-i(q,x-y)}. \tag{2.4.55}$$

The obvious λ dependence is a result of the presence of $\delta[f_b\{B^a_\mu(x);x\}]$ in the generating functional and this is one example of the δ-function's direct effect on the form of Green functions: they are, in general, gauge parameter dependent.

2.5 Exercises

(1) Repeat the derivation of Eq. (2.2.12) for $G_c(x_1,x_2,x_3)$.
(2) Prove Eq. (2.2.29).
(3) Derive Eq. (2.2.35).
(4) Prove Eq. (2.2.38).
(5) Verify Eq. (2.3.9).
(6) Verify Eqs. (2.3.13), (2.3.14).
(7) Verify Eqs. (2.3.23), (2.3.24).
(8) Verify Eqs. (2.3.26).
(9) Verify Eq. (2.3.41).
(10) Verify Eq. (2.3.56).
(11) Verify Eq. (2.3.64).
(12) Verify Eq. (2.4.44).
(13) Verify Eq. (2.4.45).
(14) Verify Eq. (2.4.46).
(15) Verify Eq. (2.4.55).

Chapter 3
Dyson-Schwinger Equations

It has long been known that, from the field equations of quantum field theory, one can derive a system of coupled integral equations interrelating all of a theory's Green functions [DY49; SC51]. This collection of a countable infinity of equations is called the complex of Dyson-Schwinger equations (DSEs). It is an intrinsically nonperturbative complex, which is vitally important in proving the renormalisability of quantum field theories, and at its simplest level the complex provides a generating tool for perturbation theory. In the context of quantum electrodynamics (QED) we will illustrate a nonperturbative derivation of two equations in this complex. The derivation of others follows the same pattern.

3.1 Photon Vacuum Polarisation

Generating Functional for QED

We begin with the action for QED with N_f flavours of electomagnetically active fermions:

$$S[A_\mu, \psi, \bar{\psi}] = \int d^4x \left[\sum_{f=1}^{N_f} \bar{\psi}^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f \right. \\ \left. -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\lambda_0} \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right]. \quad (3.1.1)$$

In this manifestly Poincaré covariant action: $\bar{\psi}^f(x)$, $\psi^f(x)$ are the elements of the Grassmann algebra that describe the fermion degrees of freedom; m_0^f are the fermions' bare masses and e_0^f their charges; and $A_\mu(x)$ describes the

gauge boson [photon] field, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{3.1.2}$$

and λ_0 the bare Lorentz gauge fixing parameter. (NB. To describe an electron the physical charge $e_f < 0$.)

The derivation of the generating functional in Eq. (2.4.50) can be employed with little change here. In fact, in this context it is actually simpler because the ghost fields decouple. Combining the procedure for fermions and gauge fields, described in Secs. 2.3, 2.4 respectively, one arrives at

$$\begin{aligned} W[J_\mu,\xi,\bar\xi] = & \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi] \\ & \times \exp\left\{ i \int d^4x \, \left[-\frac{1}{4} F^{\mu\nu}(x)\, F_{\mu\nu}(x) - \frac{1}{2\lambda_0}\, \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right.\right. \\ & \quad + \sum_{f=1}^{N_f} \bar\psi^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f \\ & \quad \left.\left. + J^\mu(x) A_\mu(x) + \bar\xi^f(x)\psi^f(x) + \bar\psi^f(x)\xi^f(x) \right] \right\} \,, \end{aligned} \tag{3.1.3}$$

where J_μ is an external source for the electromagnetic field, and ξ^f, $\bar\xi^f$ are external sources for the fermion field that, of course, are elements in the Grassmann algebra. (NB. In Abelian gauge theories there are no Gribov copies in the Lorentz gauges.)

Functional Field Equations

As described in Sec. 2.2, it is advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu,\bar\xi,\xi]$ defined via

$$W[J_\mu,\xi,\bar\xi] =: \exp\left\{ i Z[J_\mu,\xi,\bar\xi] \right\} \,. \tag{3.1.4}$$

The derivation of a DSE follows simply from the observation that the integral of a total derivative vanishes, given appropriate boundary conditions. Hence, for example,

$$\begin{aligned} 0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\, \frac{\delta}{\delta A_\mu(x)}\, \mathrm{e}^{\, i\left(S[A_\mu,\psi,\bar\psi] + \int d^4x \, \left[\overline{\psi}^f \xi^f + \bar\xi^f \psi^f + A_\mu J^\mu \right] \right)} \\ &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\, \left\{ \frac{\delta S}{\delta A_\mu(x)} + J_\mu(x) \right\} \end{aligned}$$

$$\times \exp\left\{ i\left(S[A_\mu,\psi,\bar{\psi}] + \int d^4x \, \left[\overline{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu \right] \right) \right\}$$

$$= \left\{ \frac{\delta S}{\delta A_\mu(x)} \left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\bar{\xi}}, -\frac{\delta}{i\delta\xi} \right] + J_\mu(x) \right\} W[J_\mu,\xi,\bar{\xi}] \; , \tag{3.1.5}$$

where the last line has meaning as a functional differential operator acting on the generating functional.

Differentiating Eq. (3.1.1) gives

$$\frac{\delta S}{\delta A_\mu(x)} = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0} \right) \partial_\mu \partial_\nu \right] A^\nu(x) + \sum_f e_0^f \, \overline{\psi}^f(x) \gamma_\mu \psi^f(x) \, , \tag{3.1.6}$$

so that Eq. (3.1.5) becomes

$$-J_\mu(x) = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0} \right) \partial_\mu \partial_\nu \right] \frac{\delta Z}{\delta J_\nu(x)} + \sum_f e_0^f \left(-\frac{\delta Z}{\delta \xi^f(x)} \gamma_\mu \frac{\delta Z}{\delta \bar{\xi}^f(x)} + \frac{\delta}{\delta \xi^f(x)} \left[\gamma_\mu \frac{\delta \, iZ}{\delta \bar{\xi}^f(x)} \right] \right) , \tag{3.1.7}$$

where we have divided through by $W[J_\mu,\xi,\bar{\xi}]$. Equation (3.1.7) represents a compact form of the nonperturbative equivalent of Maxwell's equations.

One-Particle-Irreducible Green Functions

The next step is to introduce the generating functional for one-particle-irreducible (1PI) Green functions: $\Gamma[A_\mu,\psi,\bar{\psi}]$, which is obtained from $Z[J_\mu,\xi,\bar{\xi}]$ via a Legendre transformation

$$Z[J_\mu,\xi,\bar{\xi}] = \Gamma[A_\mu,\psi,\bar{\psi}] + \int d^4x \, \left[\overline{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu \right] \; . \tag{3.1.8}$$

A one-particle-irreducible n-point function or "proper vertex" contains no contributions that become disconnected when a single connected m-point Green function is removed; e.g., via functional differentiation. This is equivalent to the statement that no diagram representing or contributing to a given proper vertex separates into two disconnected diagrams if only one connected propagator is cut. (A detailed explanation is provided in Ref. [IZ80], pp. 289-294.)

A simple generalisation of the analysis in Sec. 2.2 yields

$$\frac{\delta Z}{\delta J^\mu(x)} = A_\mu(x)\,,\ \frac{\delta Z}{\delta\bar\xi(x)} = \psi(x)\,,\ \frac{\delta Z}{\delta\xi(x)} = -\bar\psi(x)\,, \tag{3.1.9}$$

where here the external sources are **nonzero**. Hence Γ in Eq. (3.1.8) must satisfy

$$\frac{\delta\Gamma}{\delta A^\mu(x)} = -J_\mu(x)\,,\ \frac{\delta\Gamma}{\delta\bar\psi^f(x)} = -\xi^f(x)\,,\ \frac{\delta\Gamma}{\delta\psi^f(x)} = \bar\xi^f(x)\,. \tag{3.1.10}$$

(NB. Since the sources are not zero then, e.g.,

$$A_\rho(x) = A_\rho(x;[J_\mu,\xi,\bar\xi])\ \Rightarrow\ \frac{\delta A_\rho(x)}{\delta J^\mu(y)} \neq 0\,, \tag{3.1.11}$$

with analogous statements for the Grassmannian functional derivatives.) It is easy to see that setting $\bar\psi = 0 = \psi$ after differentiating Γ gives zero *unless* there are equal numbers of $\bar\psi$ and ψ derivatives. (This is analogous to the result for scalar fields in Eq. (2.2.38).)

Cnsider the operator and matrix product (with spinor labels r, s, t)

$$-\int d^4z\,\frac{\delta^2 Z}{\delta\xi^f_r(x)\bar\xi^h_t(z)}\,\frac{\delta^2\Gamma}{\delta\psi^h_t(z)\overline{\psi}^g_s(y)}\bigg|_{\substack{\xi=\bar\xi=0\\ \psi=\overline{\psi}=0}}\,. \tag{3.1.12}$$

Using Eqs. (3.1.9), (3.1.10), this simplifies as follows:

$$\begin{aligned} &= \int d^4z\,\frac{\delta\psi^h_t(z)}{\delta\xi^f_r(x)}\,\frac{\delta\xi^g_s(y)}{\delta\psi^h_t(z)}\bigg|_{\substack{\xi=\bar\xi=0\\ \psi=\overline{\psi}=0}} \\ &= \frac{\delta\xi^g_s(y)}{\delta\xi^f_r(x)}\bigg|_{\psi=\overline{\psi}=0} = \delta_{rs}\,\delta^{fg}\,\delta^4(x-y)\,. \end{aligned} \tag{3.1.13}$$

Returning to Eq. (3.1.7) and setting $\bar\xi = 0 = \xi$ one obtains

$$\begin{aligned} \frac{\delta\Gamma}{\delta A^\mu(x)}\bigg|_{\psi=\overline{\psi}=0} &= \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]A^\nu(x) \\ &\quad - i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu S^f(x,x;[A_\mu])\right]\,, \end{aligned} \tag{3.1.14}$$

after making the identification

$$S^f(x,y;[A_\mu]) = -\frac{\delta^2 Z}{\delta\xi^f(y)\bar{\xi}^f(x)} = \frac{\delta^2 Z}{\delta\bar{\xi}^f(x)\xi^f(y)} \text{ (no summation on } f)\,, \tag{3.1.15}$$

which is the connected Green function that describes the propagation of a fermion with flavour f in an external electromagnetic field A_μ (cf. the free fermion Green function in Eq. (2.3.56).) We note that as a direct consequence of Eq. (3.1.12) the inverse of this Green function is given by

$$S^f(x,y;[A])^{-1} = \left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(y)}\right|_{\psi=\overline{\psi}=0}\,. \tag{3.1.16}$$

It is a general property that such functional derivatives of the generating functional for 1PI Green functions are related to the associated propagator's inverse. Clearly the vacuum fermion propagator or connected fermion 2-point function is

$$S^f(x,y) := S^f(x,y;[A_\mu=0])\,. \tag{3.1.17}$$

Such vacuum Green functions are keystones in quantum field theory.

To continue, one differentiates Eq. (3.1.14) with respect to $A_\nu(y)$ and sets $J_\mu(x)=0$, which yields

$$\begin{aligned}
&\left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu=0\\ \psi=\overline{\psi}=0}} \\
&= \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y) \\
&\quad -i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu \frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1}\right]\,.
\end{aligned} \tag{3.1.18}$$

The l.h.s. is easily understood. Just as Eqs. (3.1.16), (3.1.17) define the inverse of the fermion propagator, here we have

$$(D^{-1})^{\mu\nu}(x,y) := \left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu=0\\ \psi=\overline{\psi}=0}}\,. \tag{3.1.19}$$

The r.h.s., however, must be simplified and interpreted. First observe that

$$\begin{aligned}\frac{\delta}{\delta A_\nu(y)}&\left(\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(x)}\Big|_{\psi=\overline{\psi}=0}\right)^{-1}\\ &= -\int d^4u d^4w\left(\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(w)}\Big|_{\psi=\overline{\psi}=0}\right)^{-1}\\ &\qquad\times\frac{\delta}{\delta A_\nu(y)}\frac{\delta^2\Gamma}{\delta\psi^f(u)\delta\bar\psi^f(w)}\left(\frac{\delta^2\Gamma}{\delta\psi^f(w)\delta\bar\psi^f(x)}\Big|_{\psi=\overline{\psi}=0}\right)^{-1},\end{aligned} \tag{3.1.20}$$

which is an analogue of the result for finite dimensional matrices:

$$\begin{aligned}\frac{d}{dx}\left[A(x)A^{-1}(x)=\boldsymbol{I}\right] \quad &= 0 = \quad \frac{dA(x)}{dx}A^{-1}(x)+A(x)\frac{dA^{-1}(x)}{dx}\\ &\Rightarrow \quad \frac{dA^{-1}(x)}{dx}=-A^{-1}(x)\frac{dA(x)}{dx}A^{-1}(x)\,.\end{aligned} \tag{3.1.21}$$

Equation (3.1.20) involves the 1PI 3-point function (no summation on f)

$$e_0^f\Gamma^f_\mu(x,y;z):=\frac{\delta}{\delta A_\nu(z)}\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(y)}\,. \tag{3.1.22}$$

This is the proper fermion-gauge-boson vertex. At leading order in perturbation theory

$$\Gamma^f_\nu(x,y;z)=\gamma_\nu\,\delta^4(x-z)\,\delta^4(y-z)\,, \tag{3.1.23}$$

a result which can be obtained via the explicit calculation of the functional derivatives in Eq. (3.1.22).

Defining the gauge-boson *vacuum polarisation*:

$$\Pi_{\mu\nu}(x,y)=i\sum_f(e_0^f)^2\int d^4z_1\,d^4z_2\,\mathrm{tr}\left[\gamma_\mu S^f(x,z_1)\Gamma^f_\nu(z_1,z_2;y)S^f(z_2,x)\right], \tag{3.1.24}$$

it is immediately apparent that Eq. (3.1.18) may be expressed as

$$(D^{-1})^{\mu\nu}(x,y)=\left[\partial_\rho\partial^\rho g_{\mu\nu}-\left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y)+\Pi_{\mu\nu}(x,y)\,. \tag{3.1.25}$$

In general, the gauge-boson vacuum polarisation, or "photon self-energy," describes the modification of the gauge-boson's propagation characteristics due to the presence of virtual particle-antiparticle pairs in quantum field theory. In particular, the photon vacuum polarisation is an important element in the description of process such as $\rho^0 \to e^+e^-$.

The propagator for a free gauge boson is given in Eq. (2.4.55). In the presence of interactions; i.e., for $\Pi_{\mu\nu} \neq 0$ in Eq. (3.1.25), this becomes

$$D^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2 + i\eta])}{q^2 + i\eta} \frac{1}{1 + \Pi(q^2)} - \lambda_0 \frac{q^\mu q^\nu}{(q^2 + i\eta)^2} \,, \quad (3.1.26)$$

where I have used the "Ward-Takahashi identity:"

$$q_\mu \, \Pi_{\mu\nu}(q) = 0 = \Pi_{\mu\nu}(q) \, q_\nu \,, \quad (3.1.27)$$

which means that one can write

$$\Pi^{\mu\nu}(q) = \left(-g^{\mu\nu} q^2 + q^\mu q^\nu\right) \, \Pi(q^2) \,. \quad (3.1.28)$$

$\Pi(q^2)$ is the polarisation scalar and, in QED, it is independent of the gauge parameter, λ_0. We note that $\lambda_0 = 1$ is called "Feynman gauge" and it is useful in perturbative calculations because it simplifies the $\Pi(q^2) = 0$ gauge boson propagator enormously. In nonperturbative applications, however, $\lambda_0 = 0$, "Landau gauge," is most useful because it ensures that the gauge boson propagator is itself transverse.

Ward-Takahashi identities (WTIs) are relations satisfied by n-point Green functions, relations which are an essential consequence of a theory's local gauge invariance; i.e., local current conservation. They can be proved directly from the generating functional and have physical implications. For example, Eq. (3.1.28) ensures that the photon remains massless in the presence of charged fermions.*

In the absence of external sources for fermions and gauge bosons, Eq. (3.1.24) can easily be represented in momentum space, for then the 2- and 3-point functions appearing therein must be translationally invariant and hence they can be simply expressed in terms of Fourier amplitudes; i.e., we

*A discussion of WTIs can be found in Ref. [BD64], pp. 299-303, and Ref. [IZ80], pp. 407-411; and their generalisation to non-Abelian theories as "Slavnov-Taylor" identities is described in Ref. [PT84], Chap. 2. See, also, the discussion around Eq. (3.4.11) herein.

have

$$i\Pi_{\mu\nu}(q) = -\sum_f (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^{\rm d}} {\rm tr}[(i\gamma_\mu)(iS^f(\ell))(i\Gamma^f(\ell,\ell+q))(iS(\ell+q))]\,. \tag{3.1.29}$$

It is the reduction to a single integral that makes momentum space representations most widely used in continuum calculations.

In QED the vacuum polarisation is directly related to the running coupling constant, which is a connection that makes its importance obvious. In QCD the connection is not so direct but, nevertheless, the polarisation scalar is a key component in the evaluation of the strong running coupling.

In the above analysis we saw that second derivatives of the generating functional, $\Gamma[A_\mu,\psi,\bar\psi]$, give the inverse-fermion and -photon propagators and that the third derivative gave the proper photon-fermion vertex. In general, all derivatives of this generating functional, higher than two, produce a corresponding proper vertex, where the number and type of derivatives give the number and type of proper Green functions that it can serve to connect.

3.2 Fermion Self Energy

Equation (3.1.7) is a nonperturbative generalisation of Maxwell's equation in quantum field theory. Its derivation provides the model by which one can obtain an equivalent generalisation of Dirac's equation. To this end consider that

$$\begin{aligned}
0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\frac{\delta}{\delta\bar\psi^f(x)}\,{\rm e}^{i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\,\left[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu\right]\right)} \\
&= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\left\{\frac{\delta S}{\delta\bar\psi^f(x)}+\xi^f(x)\right\} \\
&\quad \times\exp\left\{i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\,\left[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu\right]\right)\right\} \\
&= \left\{\frac{\delta S}{\delta\bar\psi^f(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar\xi},-\frac{\delta}{i\delta\xi}\right]+\eta^f(x)\right\}W[J_\mu,\xi,\bar\xi] \qquad (3.2.1)\\
&= \left[\xi^f(x)+\left(i\not\partial-m_0^f+e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)\frac{\delta}{i\delta\bar\xi^f(x)}\right]W[J_\mu,\xi,\bar\xi]. \qquad (3.2.2)
\end{aligned}$$

The last line furnishes a nonperturbative functional equivalent of Dirac's equation.

One can proceed further. A functional derivative with respect to ξ^f: $\delta/\delta\xi^f(y)$, yields

$$\delta^4(x-y)W[J_\mu] - \left(i\not{\partial} - m_0^f + e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)W[J_\mu]\,S^f(x,y;[A_\mu]) = 0\,, \tag{3.2.3}$$

after setting $\xi^f = 0 = \bar{\xi}^f$, where $W[J_\mu] := W[J_\mu, 0, 0]$ and $S(x,y;[A_\mu])$ is defined in Eq. (3.1.15). Now, using Eqs. (3.1.4), (3.1.10), this can be rewritten

$$\delta^4(x-y) - \left(i\not{\partial} - m_0^f + e_0^f\not{A}(x;[J]) + e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)\,S^f(x,y;[A_\mu]) = 0\,, \tag{3.2.4}$$

which defines the nonperturbative connected 2-point fermion Green function (This is clearly the functional equivalent of Eq. (1.4.18).)

The electromagentic four-potential vanishes in the absence of an external source; i.e., $A_\mu(x;[J=0]) = 0$, so it remains only to exhibit the content of the remaining functional differentiation in Eq. (3.2.4), which can be accomplished using Eq. (3.1.20):

$$\begin{aligned}
&\frac{\delta}{i\delta J^\mu(x)}\,S^f(x,y;[A_\mu]) \\
&\quad= \int d^4z\,\frac{\delta A_\nu(z)}{i\delta J^\mu(x)}\,\frac{\delta}{\delta A_\nu(z)}\,\left(\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(y)}\bigg|_{\psi=\overline{\psi}=0}\right)^{-1} \\
&\quad= -e_0^f\int d^4z\,d^4u\,d^4w\,\frac{\delta A_\nu(z)}{i\delta J_\mu(x)}\,S^f(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y) \\
&\quad= -e_0^f\int d^4z\,d^4u\,d^4w\,iD_{\mu\nu}(x-z)\,S^f(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y)\,,
\end{aligned} \tag{3.2.5}$$

where, in the last line, we have set $J = 0$ and used Eq. (3.1.19). Hence in the absence of external sources Eq. (3.2.4) is equivalent to

$$\begin{aligned}
\delta^4(x-y) &= \left(i\not{\partial} - m_0^f\right)\,S^f(x,y) \\
&\quad - i\,(e_0^f)^2\int d^4z\,d^4u\,d^4w\,D^{\mu\nu}(x,z)\,\gamma_\mu\,S(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y)\,.
\end{aligned} \tag{3.2.6}$$

Just as the photon vacuum polarisation was introduced to simplify, or re-express, the DSE for the gauge boson propagator, Eq. (3.1.24), one can define a fermion self-energy:

$$\Sigma^f(x,z) = i(e_0^f)^2 \int d^4u\, d^4w\; D^{\mu\nu}(x,z)\,\gamma_\mu\, S(x,u)\,\Gamma_\nu(u,w;z)\,, \qquad (3.2.7)$$

so that Eq. (3.2.6) assumes the form

$$\int d^4z\, \left[\left(i\partial\!\!\!/_x - m_0^f\right)\delta^4(x-z) - \Sigma^f(x,z)\right]\, S(z,y) = \delta^4(x-y)\,. \qquad (3.2.8)$$

Again using the property that Green functions are translationally invariant in the absence of external sources, the equation for the self-energy can be written in momentum space:

$$-i\Sigma^f(p) = (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4}\,[iD^{\mu\nu}(p-\ell)]\,[i\gamma_\mu]\,[iS^f(\ell)]\,[i\Gamma^f_\nu(\ell,p)]\,. \qquad (3.2.9)$$

In terms of the self-energy, it follows from Eq. (3.2.8) that the connected fermion 2-point function can be written in momentum space as

$$S^f(p) = \frac{1}{p\!\!\!/ - m_0^f - \Sigma^f(p) + i\eta^+}\,. \qquad (3.2.10)$$

Equation (3.2.9) is the **exact** *Gap Equation*. It describes the manner in which the propagation characteristics of a fermion moving through the ground state of QED (the QED vacuum) is altered by the repeated emission and reabsorption of virtual photons. The equation can also describe the real process of Bremsstrahlung. Furthermore, the solution of the analogous equation in QCD provides information about dynamical chiral symmetry breaking and also quark confinement. We will subsequently discuss this in detail.

3.3 Perturbation Theory

3.3.1 *Quark Self-Energy*

A key feature of strong interaction physics is dynamical chiral symmetry breaking (DCSB). In order to understand DCSB one must first come to terms with explicit chiral symmetry breaking. Consider then the DSE for

the quark self-energy in QCD:

$$-i\,\Sigma(p) = -g_0^2 \int \frac{d^4\ell}{(2\pi)^4}\, D^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a\gamma_\mu\, S(\ell)\, i\Gamma^a_\nu(\ell,p)\,, \tag{3.3.1}$$

where I have suppressed the flavour label. The form is precisely the same as that in QED, Eq. (3.2.9), with the only difference being the introduction of the colour (Gell-Mann) matrices: $\{\lambda^a; a = 1,\ldots,8\}$ at the fermion-gauge-boson vertex. The interpretation of the symbols is also analogous: $D^{\mu\nu}(\ell)$ is the connected gluon 2-point function and $\Gamma^a_\nu(\ell,\ell')$ is the proper quark-gluon vertex.

The one-loop contribution to the quark's self-energy is obtained by evaluating the r.h.s. of Eq. (3.3.1) using the free quark and gluon propagators, and the quark-gluon vertex:

$$\Gamma^{a\,(0)}_\nu(\ell,\ell') = \frac{1}{2}\lambda^a\gamma_\nu\,, \tag{3.3.2}$$

which appears to be a straightforward task. To be explicit, the goal is to calculate

$$\begin{aligned} -i\,\Sigma^{(2)}(p) \;&=\; -g_0^2 \int \frac{d^4k}{(2\pi)^4} \left(-g^{\mu\nu} + (1-\lambda_0)\frac{k^\mu k^\nu}{k^2+i\eta^+}\right) \frac{1}{k^2+i\eta^+} \\ &\quad\times \frac{i}{2}\lambda^a\gamma_\mu\, \frac{1}{\not{k}+\not{p}-m_0+i\eta^+}\, \frac{i}{2}\lambda^a\gamma_\mu\,. \end{aligned} \tag{3.3.3}$$

To proceed, first observe that Eq. (3.3.3) can be re-expressed as

$$\begin{aligned} -i\,\Sigma^{(2)}(p) = -g_0^2\, C_2(R) \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k+p)^2-m_0^2+i\eta^+}\, \frac{1}{k^2+i\eta^+} \\ \times\left\{\gamma^\mu\,(\not{k}+\not{p}+m_0)\,\gamma_\mu - (1-\lambda_0)\,(\not{k}-\not{p}+m_0) - 2\,(1-\lambda_0)\,\frac{(k,p)\not{k}}{k^2+i\eta^+}\right\}, \end{aligned} \tag{3.3.4}$$

where I have used

$$\frac{1}{2}\lambda^a\,\frac{1}{2}\lambda^a = C_2(R)\,\boldsymbol{I}_c\,;\; C_2(R) = \frac{N_c^2-1}{2N_c}\,, \tag{3.3.5}$$

with N_c the number of colours ($N_c = 3$ in QCD), and $\boldsymbol{I}_c$ is the identity matrix in colour space. Now note that

$$2\,(k,p) = [(k+p)^2 - m_0^2] - [k^2] - [p^2 - m_0^2] \tag{3.3.6}$$

and hence

$$\begin{aligned} -i\,\Sigma^{(2)}(p) &= -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+} \\ &\quad\Big\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not p-m_0) \\ &\quad+(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not k}{k^2+i\eta^+} \\ &\quad-(1-\lambda_0)\,[(k+p)^2-m_0^2]\,\frac{\not k}{k^2+i\eta^+}\Big\}. \end{aligned} \tag{3.3.7}$$

Focusing on the last term:

$$\begin{aligned} &\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\,[(k+p)^2-m_0^2]\,\frac{\not k}{k^2+i\eta^+} \\ =\;&\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i\eta^+}\,\frac{\not k}{k^2+i\eta^+}=0 \end{aligned} \tag{3.3.8}$$

because the integrand is odd under $k\to -k$, and so this term in Eq. (3.3.7) vanishes, leaving

$$\begin{aligned} -i\,\Sigma^{(2)}(p) &= -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+} \\ &\quad\Big\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not p-m_0) \\ &\quad+(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not k}{k^2+i\eta^+}\Big\}. \end{aligned} \tag{3.3.9}$$

Consider the second term:

$$(1-\lambda_0)\,(\not p-m_0)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\,.$$

In particular, focus on the behaviour of the integrand at large k^2:

$$\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\overset{k^2\to\pm\infty}{\sim}\frac{1}{(k^2-m_0^2+i\eta^+)\,(k^2+i\eta^+)}\,. \tag{3.3.10}$$

The integrand has poles in the second and fourth quadrant of the complex-k_0-plane but vanishes on any circle of radius $R\to\infty$ in this plane. That

means one may rotate the contour anticlockwise to find

$$\begin{aligned}
&\int_0^\infty dk^0 \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\
&\quad = \quad \int_0^{i\infty} dk^0 \, \frac{1}{([k^0]^2 - \vec{k}^2 - m_0^2 + i\eta^+)([k^0]^2 - \vec{k}^2 + i\eta^+)} \\
&\quad \stackrel{k^0 \to i k_4}{=} \quad i \int_0^\infty dk_4 \, \frac{1}{(-k_4^2 - \vec{k}^2 - m_0^2)\,(-k_4^2 - \vec{k}^2)} \, . \qquad (3.3.11)
\end{aligned}$$

Performing a similar analysis of the $\int_{-\infty}^0$ part, one obtains the complete result:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\
&\quad = \quad i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^\infty \frac{dk_4}{2\pi} \, \frac{1}{(-\vec{k}^2 - k_4^2 - m_0^2)\,(-\vec{k}^2 - k_4^2)} \, . \qquad (3.3.12)
\end{aligned}$$

These two steps constitute what is called a *Wick rotation*.

The integral on the r.h.s. is defined in a four-dimensional Euclidean space; i.e., $k^2 := k_1^2 + k_2^2 + k_3^2 + k_4^2 \geq 0$, with k^2 nonnegative. A general vector in this space can be written in the form:

$$(k) = |k| \, (\cos\phi \, \sin\theta \, \sin\beta, \sin\phi \, \sin\theta \, \sin\beta, \cos\theta \, \sin\beta, \cos\beta) \, ; \qquad (3.3.13)$$

i.e., using hyperspherical coordinates, and clearly $k^2 = |k|^2$. In this Euclidean space using these coordinates the four-vector measure factor is

$$\begin{aligned}
&\int d_E^4 k \, f(k_1, \ldots, k_4) \\
&\quad = \quad \tfrac{1}{2} \int_0^\infty dk^2 \, k^2 \int_0^\pi d\beta \, sin^2\beta \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, f(k, \beta, \theta, \phi) \, . \qquad (3.3.14)
\end{aligned}$$

Returning to Eq. (3.3.10) and making use of the material just introduced, the large k^2 behaviour of the integral can be determined via

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k+p)^2 - m_0^2 + i\eta^+} \, \frac{1}{k^2 + i\eta^+} \\
&\quad \approx \quad \frac{i}{16\pi^2} \int_0^\infty dk^2 \frac{1}{(k^2 + m_0^2)}
\end{aligned}$$

$$= \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} dx \, \frac{1}{x+m_0^2}$$
$$= \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \ln\left(1+\Lambda^2/m_0^2\right) \to \infty\,; \qquad (3.3.15)$$

i.e., after all this work, the result is meaningless: the one-loop contribution to the quark's self-energy is divergent!

Such "ultraviolet" divergences, and others which are more complicated, arise whenever loops appear in perturbation theory. (The others include "infrared" divergences associated with the gluons' masslessness; e.g., consider what would happen in Eq. (3.3.15) with $m_0 \to 0$.) In a *renormalisable* quantum field theory there exists a well-defined set of rules that can be used to render perturbation theory sensible. First, however, one must *regularise* the theory; i.e., introduce a cutoff, or use some other means, to make finite every integral that appears. Then each step in the calculation of an observable is rigorously sensible. Renormalisation follows; i.e, the absorption of divergences, and the redefinition of couplings and masses, so that finally one arrives at $\mathcal{S}$-matrix amplitudes that are finite and physically meaningful.

The regularisation procedure must preserve the Ward-Takahashi identities (the Slavnov-Taylor identities in QCD) because they are crucial in proving that a theory can be sensibly renormalised. A theory is called renormalisable if, and only if, the number of different types of divergent integral is finite. In that case only a finite number of masses and coupling constants need to be renormalised; i.e., *a priori* the theory has only a finite number of undetermined parameters that must be fixed through comparison with experiments.

3.3.2 *Dimensional Regularisation*

The Pauli-Villars regularisation prescription is favoured in QED and that is described, for example, in Ref. [IZ80]. In perturbative QCD, however, dimensional regularisation is the most commonly used procedure and we will introduce that herein.

The key to the method lies in giving meaning to the divergent integrals by changing the dimension of spacetime. Returning to the exemplar, Eq. (3.3.10), this means we consider

$$T = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k+p)^2 - m_0^2 + i\eta^+} \frac{1}{k^2 + i\eta^+}\,, \qquad (3.3.16)$$

where D is the dimension of spacetime and is not necessarily four.

Observe that

$$\frac{1}{a^\alpha\, b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 dx\, \frac{x^{\alpha-1}\,(1-x)^{\beta-1}}{[a\,x + b\,(1-x)]^{\alpha+\beta}}\,, \tag{3.3.17}$$

where $\Gamma(x)$ is the gamma-function: $\Gamma(n+1) = n!$. This is an example of what is commonly called "Feynman's parametrisation," and one can use it to write

$$T = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{[(k-xp)^2 - m_0^2(1-x) + p^2 x(1-x) + i\eta^+]^2}\,. \tag{3.3.18}$$

The momentum integral is well-defined for $D = 1, 2, 3$ but, as we have seen, not for $D = 4$. One proceeds under the assumption that D is such that the integral is convergent for then a shift of variables is permitted:

$$T \overset{k\to k-xp}{=} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{[k^2 - a^2 + i\eta^+]^2}\,, \tag{3.3.19}$$

where $a^2 = m_0^2(1-x) - p^2 x(1-x)$.

For illustrative purposes we will consider a generalisation of the momentum integral:

$$I_n \;=\; \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{[k^2 - a^2 + i\eta^+]^n}\,, \tag{3.3.20}$$

and perform a Wick rotation to obtain

$$I_n \;=\; \frac{i}{(2\pi)^D}\,(-1)^n \int d^D k\, \frac{1}{[k^2 + a^2]^n}\,. \tag{3.3.21}$$

The integrand has an $O(D)$ spherical symmetry and therefore the angular integrals can be performed:

$$S_D := \int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{3.3.22}$$

Clearly, $S_4 = 2\pi^2$, as we saw in Eq. (3.3.15). Hence

$$I_n \;=\; i\frac{(-1)^n}{2^{D-1}\pi^{D/2}}\, \frac{1}{\Gamma(D/2)} \int_0^\infty dk\, \frac{k^{D-1}}{(k^2+a^2)^n}\,. \tag{3.3.23}$$

Writing $D = 4 + 2\epsilon$ one arrives at

$$I_n = \frac{i}{(4\pi)^2} (-a^2)^{2-n} \left(\frac{a^2}{4\pi}\right)^{\epsilon} \frac{\Gamma(n-2-\epsilon)}{\Gamma(n)} . \tag{3.3.24}$$

(NB. Every step is rigorously justified as long as $2n > D$.) The important point for continuing with this procedure is that the analytic continuation of $\Gamma(x)$ is unique and that means one may use Eq. (3.3.24) as the *definition* of I_n whenever the integral is ill-defined. We note that $D = 4$ is recovered via the limit $\epsilon \to 0^-$ and the divergence of the integral for $n = 2$ in this case is encoded in

$$\Gamma(n-2-\epsilon) = \Gamma(-\epsilon) = \frac{1}{-\epsilon} - \gamma_E + \mathrm{O}(\epsilon) ; \tag{3.3.25}$$

i.e., in the pole in the gamma-function. (γ_E is the Euler constant.)

Substituting Eq. (3.3.24) in Eq. (3.3.19) and setting $n = 2$ yields

$$T = (g\nu^{\epsilon})^2 \frac{i}{(4\pi)^2} \frac{\Gamma(-\epsilon)}{(4\pi)^{\epsilon}} \int_0^1 dx \left[\frac{m_0^2}{\nu^2}(1-x) - \frac{p^2}{\nu^2} x(1-x)\right]^{\epsilon} , \tag{3.3.26}$$

wherein we have employed a nugatory transformation to introduce the mass-scale ν. It is the limit $\epsilon \to 0^-$ that is of interest, in which case it follows that ($x^{\epsilon} = \exp(\epsilon \ln x) \approx 1 + \epsilon \ln x$)

$$\begin{aligned} T &= (g\nu^{\epsilon})^2 \frac{i}{(4\pi)^2} \\ &\quad \times \left\{ -\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \int_0^1 dx \, \ln\left[\frac{m_0^2}{\nu^2}(1-x) - \frac{p^2}{\nu^2} x(1-x)\right] \right\} \\ &= (g\nu^{\epsilon})^2 \frac{i}{(4\pi)^2} \\ &\quad \times \left\{ -\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 2 - \ln\frac{m_0^2}{\nu^2} - \left(1 - \frac{m_0^2}{p^2}\right) \ln\left[1 - \frac{p^2}{m_0^2}\right] \right\} . \end{aligned} \tag{3.3.27}$$

It is important to understand the physical content of Eq. (3.3.27). While it is only one part of the gluon's contribution to the quark's self-energy, many of its properties hold generally.

(1) Observe that $T(p^2)$ is well-defined for $p^2 < m_0^2$; i.e., for all spacelike momenta and for a small domain of timelike momenta. However, at $p^2 = m_0^2$, $T(p^2)$ exhibits a ln-branch-point and hence $T(p^2)$ acquires

an imaginary part for $p^2 > m_0^2$. This imaginary part describes the real, physical process by which a quark emits a massless gluon; i.e., gluon Bremsstrahlung. In QCD this is one element in the collection of processes referred to as "quark fragmentation."

(2) The mass-scale, ν, introduced in Eq. (3.3.26), which is a theoretical artifice, does not affect the position of the branch-point. That is very good because that branch-point is associated with observable phenomena. While it may appear that ν affects the magnitude of physical cross sections because it modifies the coupling, that is not really the case: in going to $D = 2n - \epsilon$ dimensions the coupling constant, which was dimensionless for $D = 4$, has acquired a mass dimension and so the physical, dimensionless coupling constant is $\alpha := (g\nu^\epsilon)^2/(4\pi)$. It is this dependence on ν that opens the door to the generation of a *running coupling constant* and *running masses*, which are a hallmark of quantum field theory.

(3) A number of constants have appeared in $T(p^2)$. These are irrelevant because they are eliminated in the renormalisation procedure. NB. So far we have only regularised the expression; renormalisation is another step.

(4) It is apparent that dimensional regularisation gives meaning to divergent integrals without introducing new couplings or new fields. That is a benefit. The cost is that while $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is well-defined with particular properties for $D = 4$, a generalisation to $D \neq 4$ is difficult and hence so is the study of chiral symmetry.

D-dimensional Dirac algebra

When one employs dimensional regularisation all algebraic manipulations must be performed before the integrals are evaluated, and that includes the Dirac algebra. The Clifford algebra is unchanged in D-dimensions

$$\{\gamma^\mu\,,\,\gamma^\nu\} = 2\,g^{\mu\nu}\,\mathbf{1}_D\,;\;\mu,\nu = 0,1,\ldots,D-1\,, \tag{3.3.28}$$

where $\mathbf{1}_D$ is the $D \times D$-dimensional unit matrix, but now

$$g_{\mu\nu}\,g^{\mu\nu} = D \tag{3.3.29}$$

and hence

$$\gamma_\mu\,\gamma^\nu = D\,\mathbf{1}_D\,, \tag{3.3.30}$$

$$\gamma_\mu\,\gamma^\nu\,\gamma^\mu = (2-D)\,\gamma^\nu\,, \tag{3.3.31}$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4\, g^{\nu\lambda} \mathbf{1}_D + (D-4)\, \gamma^\nu \gamma^\lambda \,, \tag{3.3.32}$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma^\mu = -2\, \gamma^\rho \gamma^\lambda \gamma^\nu + (4-D)\, \gamma^\nu \gamma^\lambda \gamma^\rho \,. \tag{3.3.33}$$

It is also necessary to evaluate traces of products of Dirac matrices. For a D-dimensional space, with D even, the only irreducible representation of the Clifford algebra, Eq. (3.3.28), has dimension $f(D) = 2^{D/2}$. In any calculation it is the (anti-)commutation of Dirac matrices that leads to physically important factors associated with the dimension of spacetime, while $f(D)$ always appears simply as a common multiplicative factor. Hence one can just set

$$f(D) \equiv 4 \tag{3.3.34}$$

in all calculations. Any other prescription merely leads to constant terms of the type encountered above; e.g., γ_E, which are eliminated in renormalisation.

The D-dimensional generalisation of γ_5 is a more intricate problem. However, we will not use it herein and hence omit that discussion.

Observations on the appearance of divergences

Consider a general Lagrangian density:

$$L(x) = L_0(x) + \sum_i g_i L_i(x) \,, \tag{3.3.35}$$

where L_0 is the sum of the free-particle Lagrangian densities and $L_i(x)$ represents the interaction terms with the coupling constants, g_i, written explicitly. Assume that $L_i(x)$ has f_i fermion fields (f_i must be even since fermion fields always appear in the pairs $\bar{\psi}$, ψ), b_i boson fields and n_i^∂ derivatives. The action

$$S = \int d^4x\, L(x) \tag{3.3.36}$$

must be a dimensionless scalar and therefore $L(x)$ must have mass-dimension M^4. Clearly a derivative operator has dimension M. Hence, looking at the free-particle Lagrangian densities, it is plain that each fermion field has dimension $M^{3/2}$ and each boson field, dimension M^1. It follows that a coupling constant multiplying the interaction Lagrangian density $L_i(x)$ must have mass-dimension

$$[g_i] = M^{4-d_i} \,,\; d_i = \frac{3}{2} f_i + b_i + n_i^\partial \,. \tag{3.3.37}$$

It is a fundamentally important fact in quantum field theory that even if there is just one coupling constant for which

$$[g_i] < 0 \tag{3.3.38}$$

then the theory possesses infinitely many different types of divergences and hence cannot be rendered finite through a finite number of renormalisations. This defines the term nonrenormalisable.

In the past nonrenormalisable theories were rejected as having no predictive power: if one has to remove infinitely many infinite constants before one can define a result then that result cannot be meaningful. However, the modern view is different. For example, chiral perturbation theory is a nonrenormalisable "effective field theory." However, at any finite order in perturbation theory there is only a finite number of undetermined constants: 2 at leading order; 6 more, making 8 in total, when one-loop effects are considered; and more than 140 new terms when two-loop effects are admitted. Nevertheless, so long as there is a domain in the space of physical observables on which the one-loop corrected Lagrangian density can be assumed to be a good approximation, and in this domain there is more data that can be described than there are undetermined constants, then this "effective theory" can be useful as a tool for correlating observables and elucidating the symmetries that underly the general pattern of hadronic behaviour.

3.3.3 *Regularised Quark Self-Energy*

We can now return and re-express Eq. (3.3.9):

$$\begin{aligned} -i\,\Sigma^{(2)}(p) \;=\; & -(g_0\nu^\epsilon)^2\,C_2(R)\int\frac{d^Dk}{(2\pi)^4}\,\frac{1}{\nu^{2\epsilon}}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+} \\ & \Big\{\gamma^\mu\,(\not{k}+\not{p}+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not{p}-m_0) \\ & +(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not{k}}{k^2+i\eta^+}\Big\}\,. \end{aligned} \tag{3.3.39}$$

It can be separated into a sum of two terms, each proportional to a different Dirac structure:

$$\Sigma(\not{p}) = \Sigma_V(p^2)\,\not{p} + \Sigma_S(p^2)\,\mathbf{1}_D\,, \tag{3.3.40}$$

that can be obtained via trace projection:

$$\Sigma_V(p^2) = \frac{1}{f(D)}\,\frac{1}{p^2}\,\mathrm{tr}_D\left[\not{p}\Sigma(\not{p})\right]\,,\;\Sigma_S(p^2) = \frac{1}{f(D)}\,\mathrm{tr}_D\left[\Sigma(\not{p})\right]\,. \quad (3.3.41)$$

These are the vector and scalar parts of the dressed-quark self-energy, and they are easily found to be

$$\begin{aligned} p^2\,\Sigma_V(p^2) &= -i\,(g_0\nu^\epsilon)^2\,C_2(R)\int\frac{d^Dk}{(2\pi)^4}\,\frac{1}{\nu^{2\epsilon}}\,\left\{\frac{1}{(k+p)^2-m_0^2+i\eta^+}\right. \\ &\times\frac{1}{[k^2+i\eta^+]}\,\left[(2-D)(p^2+p_\mu k^\mu)\right. \\ &\left.\left.+(1-\lambda_0)p^2+(1-\lambda_0)(p^2-m_0^2)\frac{p_\mu k^\mu}{k^2+i\eta^+}\right]\right\}. \end{aligned} \quad (3.3.42)$$

$$\begin{aligned} &\Sigma_S(p^2) \\ &\quad = -i\,(g_0\nu^\epsilon)^2\,C_2(R)\int\frac{d^Dk}{(2\pi)^4}\,\frac{1}{\nu^{2\epsilon}}\,\frac{m_0(D-1+\lambda_0)}{[(k+p)^2-m_0^2+i\eta^+]\,[k^2+i\eta^+]}\,. \end{aligned} \quad (3.3.43)$$

(NB. As promised, the factor of $f(D)$ has cancelled.)

These equations involve integrals of the general form

$$I(\alpha,\beta;p^2,m^2) = \int\frac{d^Dk}{(2\pi)^4}\,\frac{1}{\nu^{2\epsilon}}\,\frac{1}{[(k+p)^2-m^2+i\eta^+]^\alpha\,[k^2+i\eta^+]^\beta}\,, \quad (3.3.44)$$

$$J^\mu(\alpha,\beta;p^2,m^2) = \int\frac{d^Dk}{(2\pi)^4}\,\frac{1}{\nu^{2\epsilon}}\,\frac{k^\mu}{[(k+p)^2-m^2+i\eta^+]^\alpha\,[k^2+i\eta^+]^\beta}\,, \quad (3.3.45)$$

the first of which we have already encountered in Sec. 3.3.2. The general results are ($D = 4+2\epsilon$)

$$\begin{aligned} &I(\alpha,\beta;p^2,m^2) \\ &\quad = \frac{i}{(4\pi)^2}\left(\frac{1}{p^2}\right)^{\alpha+\beta-2}\,\frac{\Gamma(\alpha+\beta-2-\epsilon)\,\Gamma(2+\epsilon-\beta)}{\Gamma(\alpha)\,\Gamma(2+\epsilon)} \\ &\qquad\times\left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon\,\left(1-\frac{m^2}{p^2}\right)^{2+\epsilon-\alpha-\beta} \\ &\qquad\times{}_2F_1(\alpha+\beta-2-\epsilon,2+\epsilon-\beta,2+\epsilon;\frac{1}{1-(m^2/p^2)})\,, \end{aligned} \quad (3.3.46)$$

$$
\begin{aligned}
p^\mu\, J(\alpha,\beta;p^2,m^2) &:= J^\mu(\alpha,\beta;p^2,m^2) \\
&= p^\mu\, \frac{i}{(4\pi)^2}\left(\frac{1}{p^2}\right)^{\alpha+\beta-2} \frac{\Gamma(\alpha+\beta-2-\epsilon)\,\Gamma(3+\epsilon-\beta)}{\Gamma(\alpha)\,\Gamma(3+\epsilon)} \\
&\quad \times \left(-\frac{p^2}{4\pi\nu^2}\right)^{\epsilon} \left(1-\frac{m^2}{p^2}\right)^{2+\epsilon-\alpha-\beta} \\
&\quad \times {}_2F_1(\alpha+\beta-2-\epsilon, 3+\epsilon-\beta, 3+\epsilon; \frac{1}{1-(m^2/p^2)})\,, \quad (3.3.47)
\end{aligned}
$$

where ${}_2F_1(a,b,c;z)$ is the hypergeometric function.

Again returning to Eqs. (3.3.42), (3.3.43) it is plain that

$$
\begin{aligned}
\Sigma_V(p^2) &= -i\,(g_0\nu^\epsilon)^2\, C_2(R) \left\{ [2(1+\epsilon)\, J(1,1) - (1+2\epsilon)\, I(1,1)] \right. \\
&\quad \left. -(p^2-m_0^2)\, J(1,2) + \lambda_0 \left[(p^2-m_0^2)\, J(1,2) - I(1,1)\right] \right\}, \quad (3.3.48) \\
\Sigma_S(p^2) &= -i\,(g_0\nu^\epsilon)^2\, C_2(R)\, m_0\, (3+\lambda_0+2\epsilon)\, I(1,1;p^2,m^2)\,, \quad (3.3.49)
\end{aligned}
$$

where I have omitted the (p^2, m_0^2) component in the arguments of I, J. The integrals required explicitly are

$$
\begin{aligned}
I(1,1) = \frac{i}{(4\pi)^2} &\left\{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(-\frac{p^2}{\nu^2}\right) + 2 \right. \\
&\left. -\frac{m^2}{p^2}\ln\left(-\frac{m^2}{p^2}\right) - \left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{m^2}{p^2}\right) \right\}, \quad (3.3.50)
\end{aligned}
$$

$$
\begin{aligned}
J(1,1) = \frac{i}{(4\pi)^2}\,\frac{1}{2} &\left\{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(-\frac{p^2}{\nu^2}\right) + 2 \right. \\
&-\frac{m^2}{p^2}\left(2-\frac{m^2}{p^2}\right)\ln\left(-\frac{m^2}{p^2}\right) \\
&\left. -\left(1-2\frac{m^2}{p^2}+\left[\frac{m^2}{p^2}\right]^2\right)\ln\left(1-\frac{m^2}{p^2}\right) - \frac{m^2}{p^2} \right\}, \quad (3.3.51)
\end{aligned}
$$

$$
J(1,2) = \frac{i}{(4\pi)^2}\,\frac{1}{p^2}\left\{ -\frac{m^2}{p^2}\ln\left(-\frac{m^2}{p^2}\right) + \frac{m^2}{p^2}\ln\left(1-\frac{m^2}{p^2}\right) + 1 \right\}. \quad (3.3.52)
$$

Using these expressions it is straightforward to show that the λ_0-independent term in Eq. (3.3.48) is identically zero in $D = 4$ dimensions; i.e., for $\epsilon \to 0^-$, and hence

$$\begin{aligned}\Sigma_V(p^2) &= \lambda_0 \frac{(g\nu^\epsilon)^2}{(4\pi)^2} C_2(R) \left\{ \frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln \frac{m_0^2}{\nu^2} \right. \\ &\left. -1 - \frac{m_0^2}{p^2} + \left(1 - \frac{m_0^4}{p^4}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} . \qquad (3.3.53)\end{aligned}$$

It is obvious that in Landau gauge, $\lambda_0 = 0$, and in four spacetime dimensions one has $\Sigma_V(p^2) \equiv 0$, at this order. The scalar piece of the quark's self-energy is also found easily:

$$\begin{aligned}\Sigma_S(p^2) &= m_0 \frac{(g\nu^\epsilon)^2}{(4\pi)^2} C_2(R) \left\{ -(3+\lambda_0) \left[\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln \frac{m_0^2}{\nu^2} \right] \right. \\ &\left. +2(2+\lambda_0) - (3+\lambda_0) \left(1 - \frac{m_0^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m_0^2}\right) \right\} . \qquad (3.3.54)\end{aligned}$$

Note that in Yennie gauge, $\lambda_0 = -3$,and in four dimensions, the scalar piece of the self-energy is momentum-independent, at this order.

We now have the complete regularised dressed-quark self-energy at one-loop order in perturbation theory and its structure is precisely as I described in Sec. 3.3.2. Renormalisation must follow.

One final observation: the scalar piece of the self-energy is proportional to the bare current-quark mass, m_0. That is true at every order in perturbation theory. Clearly then

$$\lim_{m_0 \to 0} \Sigma_S(p^2, m_0^2) = 0 \qquad (3.3.55)$$

and hence dynamical chiral symmetry breaking is impossible in perturbation theory.

3.4 Renormalised Quark Self-Energy

Hitherto we have illustrated the manner in which dimensional regularisation is employed to give sense to the divergent integrals that appear in the perturbative calculation of matrix elements in quantum field theory. It is now necessary to renormalise the theory; i.e., to provide a well-defined prescription for the elimination of all those parts in the calculated matrix element that express the divergences and thereby obtain finite results for

Green functions in the limit $\epsilon \to 0^-$ (or with the removal of whatever other parameter has been used to regularise the divergences).

3.4.1 *Renormalised Lagrangian*

The bare QCD Lagrangian density is

$$\begin{aligned} L(x) &= -\frac{1}{2}\,\partial_\mu B^a_\nu(x)[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)] - \frac{1}{2\lambda}\partial_\mu B^a_\nu(x)\partial_\mu B^a_\nu(x) \\ &\quad -\frac{1}{2}\,g\,f_{abc}\,[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)]B^b_\mu(x)B^c_\nu(x) \\ &\quad -\frac{1}{4}\,g^2 f_{abc}f_{ade}\,B^b_\mu(x)B^c_\nu(x)B^{\mu d}(x)B^{\nu e}(x) \\ &\quad -\partial_\mu\bar\phi^a(x)\partial^\mu\phi^a(x) + g\,f_{abc}\,\partial_\mu\bar\phi^a(x)\,\phi^b(x)B^{\mu c}(x) \\ &\quad +\bar q^f(x)i\not\partial q^f(x) - m^f\,\bar q^f(x)q^f(x) + \frac{1}{2}\,g\,\bar q^f(x)\lambda^a\not B^a(x)q^f(x)\,, \end{aligned} \tag{3.4.1}$$

where: $B^a_\mu(x)$ are the gluon fields, with the colour label $a = 1, \ldots, 8$; $\bar\phi^a(x)$, $\phi^a(x)$ are the (Grassmann) ghost fields; $\bar q^f(x)$, $q^f(x)$ are the (Grassmann) quark fields, with the flavour label $f = u, d, s, c, b, t$; and g, m^f, λ are, respectively, the coupling, mass and gauge fixing parameter.†

The elimination of the divergent parts in the expression for a Green function can be achieved by adding "counterterms" to the bare QCD Lagrangian density, one for each different type of divergence in the theory; i.e., one considers the renormalised Lagrangian density

$$L_R(x) := L(x) + L_c(x)\,, \tag{3.4.2}$$

with

$$\begin{aligned} L_c(x) &= C_{3YM}\,\frac{1}{2}\,\partial_\mu B^a_\nu(x)[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)] \\ &\quad +C_6\frac{1}{2\lambda}\partial_\mu B^a_\nu(x)\partial_\mu B^a_\nu(x) \\ &\quad +C_{1YM}\,\frac{1}{2}\,g\,f_{abc}\,[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)]B^b_\mu(x)B^c_\nu(x) \\ &\quad +C_5\,\frac{1}{4}\,g^2 f_{abc}f_{ade}\,B^b_\mu(x)B^c_\nu(x)B^{\mu d}(x)B^{\nu e}(x) \\ &\quad +\tilde C_3\,\partial_\mu\bar\phi^a(x)\partial^\mu\phi^a(x) - \tilde C_1 g\,f_{abc}\,\partial_\mu\bar\phi^a(x)\,\phi^b(x)B^{\mu c}(x) \end{aligned}$$

†NB. The QED Lagrangian density is immediately obtained by setting $f_{abc} \equiv 0$. It is clearly the non-Abelian nature of the gauge group, $SU(N_c)$, that generates the gluon self-couplings, the triple-gluon and four-gluon vertices, and the ghost-gluon interaction.

$$-C_{2F}\,\bar{q}^f(x) i \not{\partial} q^f(x) + C_4\, m^f \bar{q}^f(x) q^f(x)$$
$$-C_{1F}\,\frac{1}{2}\, g\, \bar{q}^f(x) \lambda^a \not{B}^a(x) q^f(x)\,. \qquad (3.4.3)$$

To prove the renormalisability of QCD one must establish that the coefficients, C_i, each understood as a power series in g^2, are the only additional terms necessary to remove all the ultraviolet divergences in the theory at every order in the perturbative expansion.

In the example of Sec. 3.3.2 we illustrated that the divergent terms in the regularised self-energy are proportional to g^2. This is a general property and hence all of the coefficients C_i begin with a g^2 term. The C_i-dependent terms can be treated in exactly the same way as the terms in the original Lagrangian density and they yield corrections to the expressions we have already derived that begin with an order-g^2 term. Returning to the example of the dressed-quark self-energy this means that we have an additional contribution: $\Sigma^{(2)} \to \Sigma^{(2)} + \Delta\Sigma^{(2)}$, with

$$\Delta\Sigma^{(2)}(\not{p}) = C_{2F}\, \not{p} - C_4\, m\,, \qquad (3.4.4)$$

and one can choose C_{2F}, C_4 such that the total self-energy; i.e., the sum, is finite.

The renormalisation constants are introduced as follows:[‡]

$$Z_i := 1 - C_i \qquad (3.4.5)$$

so that Eq. (3.4.3) becomes

$$\begin{aligned} L_R(x) \;=\; & -\frac{Z_{3YM}}{2}\,\partial_\mu B^a_\nu(x)[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)] \\ & -\frac{Z_6}{2\lambda}\,\partial_\mu B^a_\nu(x)\partial_\mu B^a_\nu(x) \\ & -\frac{Z_{1YM}}{2}\, g\, f_{abc}\,[\partial^\mu B^{\nu a}(x) - \partial^\nu B^{\mu a}(x)] B^b_\mu(x) B^c_\nu(x) \\ & -\frac{Z_5}{4}\, g^2 f_{abc} f_{ade}\, B^b_\mu(x) B^c_\nu(x) B^{\mu d}(x) B^{\nu e}(x) \\ & -\tilde{Z}_3\,\partial_\mu \bar{\phi}^a(x)\partial^\mu \phi^a(x) + \tilde{Z}_1\, g\, f_{abc}\,\partial_\mu \bar{\phi}^a(x)\,\phi^b(x) B^{\mu c}(x) \\ & +Z_{2F}\,\bar{q}^f(x) i \not{\partial} q^f(x) - Z_4\, m^f \bar{q}^f(x) q^f(x) \end{aligned}$$

[‡]NB. We have implicitly assumed that the renormalisation counterterms, and hence the renormalisation constants, are flavour independent. It is always possible to choose prescriptions such that this is so.

$$+\frac{Z_{1F}}{2}\, g\, \bar{q}^f(x)\lambda^a \not{B}^a(x) q^f(x)\,. \tag{3.4.6}$$

We now introduce the bare fields, coupling constants, masses and gauge fixing parameter:

$$\begin{aligned}
&B_0^{\mu a}(x) := Z_{3YM}^{1/2}\, B^{\mu a}(x)\,, && q_0^f(x) := Z_{2F}^{1/2}\, q^f(x)\,,\\
&\phi_0^a(x) := \tilde{Z}_3^{1/2}\, \phi^a(x)\,, && \bar{\phi}_0^a(x) := \tilde{Z}_3^{1/2}\, \bar{\phi}^a(x)\,,\\
&g_{0YM} := Z_{1YM}\, Z_{3YM}^{-3/2}\, g\,, && \tilde{g}_0 := \tilde{Z}_1\, \tilde{Z}_3^{-1}\, Z_{3YM}^{-1/2}\,,\\
&g_{0F} := Z_{1F}\, Z_{3YM}^{-1/2}\, Z_{2F}^{-1} g\,, && g_{05} := Z_5^{1/2}\, Z_{3YM}^{-1}\, g\\
&m_0^f := Z_4\, Z_{2F}^{-1}\, m^f\,, && \lambda_0 := Z_6^{-1}\, Z_{3YM}\, \lambda\,.
\end{aligned} \tag{3.4.7}$$

The fields and couplings on the r.h.s. of these definitions are called *renormalised*, and the couplings are finite and the fields produce Green functions that are finite even in $D = 4$ dimensions. (NB. All these quantities are defined in $D = 4 + 2\epsilon$-dimensional space. Hence one has for the Lagrangian density: $[L(x)] = M^D$, and the field and coupling dimensions are

$$\begin{aligned}
&[q(x)] = [\bar{q}(x)] = M^{3/2+\epsilon}\,, && [B^\mu(x)] = M^{1+\epsilon}\,,\\
&[\phi(x)] = [\bar{\phi}(x)] = M^{1+\epsilon}\,, && [g] = M^{-\epsilon}\,,\\
&[\lambda] = M^0\,, && [m] = M^1\,.)
\end{aligned} \tag{3.4.8}$$

The renormalised Lagrangian density can be rewritten in terms of the bare quantities:

$$\begin{aligned}
L_R(x) \;=\;& -\frac{1}{2}\,\partial_\mu B_{0\nu}^a(x)[\partial^\mu B_0^{\nu a}(x) - \partial^\nu B_0^{\mu a}(x)] - \frac{1}{2\lambda_0}\partial_\mu B_{0\nu}^a(x)\partial_\mu B_{0\nu}^a(x)\\
&-\frac{1}{2}\, g_{0YM}\, f_{abc}\,[\partial^\mu B_0^{\nu a}(x) - \partial^\nu B_0^{\mu a}(x)] B_{0\mu}^a(x) B_{0\nu}^a(x)\\
&-\frac{1}{4}\, g_{05}^2 f_{abc} f_{ade}\, B_{0\mu}^b(x) B_{0\nu}^c(x) B_0^{\mu d}(x) B_0^{\nu e}(x)\\
&-\partial_\mu \bar{\phi}_0^a(x)\partial^\mu \phi_0^a(x) + \tilde{g}_0\, f_{abc}\, \partial_\mu \bar{\phi}_0^a(x)\, \phi_0^b(x) B_0^{\mu c}(x)\\
&+\bar{q}_0^f(x) i \not{\partial} q_0^f(x) - m_0^f \bar{q}_0^f(x) q_0^f(x)\\
&+\frac{1}{2}\, g_{0F}\, \bar{q}_0^f(x)\lambda^a \not{B}_0^a(x) q_0^f(x)\,.
\end{aligned} \tag{3.4.9}$$

It is apparent that the couplings in the renormalised Lagrangian density are different and hence $L_R(x)$ is not invariant under local gauge transformations (more properly, BRST transformations) unless

$$g_{0YM} = \tilde{g}_0 = g_{0F} = g_{05} = g_0\,. \tag{3.4.10}$$

Therefore, if the renormalisation procedure is to preserve the character of the gauge theory, the renormalisation constants cannot be completely arbitrary but must satisfy the following "Slavnov-Taylor" identities:

$$\begin{aligned} g_{0YM} = \tilde{g}_0 \quad &\Rightarrow \quad \frac{Z_{3YM}}{Z_{1YM}} = \frac{\tilde{Z}_3}{\tilde{Z}_1}\,, \\ g_{0YM} = g_{0F} \quad &\Rightarrow \quad \frac{Z_{3YM}}{Z_{1YM}} = \frac{Z_{2F}}{Z_{1F}}\,, \\ g_{0YM} = g_{05} \quad &\Rightarrow \quad Z_5 = \frac{Z_{1YM}^2}{Z_{3YM}}\,. \end{aligned} \tag{3.4.11}$$

In QED the second of these equations becomes the Ward-Takahashi identity: $Z_{1F} = Z_{2F}$.

3.4.2 *Renormalisation Schemes*

At this point we can immediately write an expression for the renormalised dressed-quark self-energy:

$$\Sigma_R^{(2)}(\not{p}) = \left(\Sigma_V(p^2) + C_{2F}\right)\, \not{p} + \Sigma_S(p^2) - C_4\, m\,. \tag{3.4.12}$$

The subtraction constants are not yet determined and there are many ways one may choose them in order to eliminate the divergent parts of bare Green functions.

<u>*Minimal Subtraction*</u>
In the minimal subtraction (MS) scheme one defines a dimensionless coupling

$$\alpha := \frac{(g\nu^\epsilon)^2}{4\pi} \tag{3.4.13}$$

and considers each counterterm as a power series in α with the form

$$C_i = \sum_{j=1}^{\infty} \sum_{k=1}^{j} C_{i,k}^{(2j)}\, \frac{1}{\epsilon^k}\, \left(\frac{\alpha}{\pi}\right)^j\,, \tag{3.4.14}$$

where the coefficients in the expansion may, at most, depend on the gauge parameter, λ.

Using Eqs. (3.3.53), (3.3.54) and (3.4.12) we have

$$\Sigma_R^{(2)}(\not{p}) \quad = \quad \not{p} \left\{ \frac{\alpha}{\pi}\, \lambda\, \frac{1}{4}\, C_2(R)\, \left[\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln \frac{m^2}{\nu^2} \right.\right.$$

$$
\begin{aligned}
&-1-\frac{m^2}{p^2}+\left(1-\frac{m^4}{p^4}\right)\ln\left(1-\frac{p^2}{m^2}\right)\Big]+C_{2F}\Big\}\\
&+m\,\frac{\alpha}{\pi}\,\frac{1}{4}\,C_2(R)\,\Big\{-(3+\lambda)\left[\frac{1}{\epsilon}-\ln 4\pi+\gamma_E+\ln\frac{m^2}{\nu^2}\right]\\
&+2(2+\lambda)-(3+\lambda)\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)-C_4\Big\}\,.
\end{aligned}
\tag{3.4.15}
$$

Now one chooses C_{2F}, C_4 such that they cancel the $1/\epsilon$ terms in this equation and therefore, at one-loop level,

$$
Z_{2F}=1-C_{2F} \;=\; 1+\frac{\alpha}{\pi}\,\lambda\,\frac{1}{4}\,C_2(R)\,\frac{1}{\epsilon}\,, \tag{3.4.16}
$$

$$
Z_4=1-C_4 \;=\; 1+\frac{\alpha}{\pi}\,(3+\lambda)\,\frac{1}{4}\,C_2(R)\,\frac{1}{\epsilon}\,, \tag{3.4.17}
$$

and hence Eq. (3.4.15) becomes

$$
\begin{aligned}
\Sigma_R^{(2)}(\not{p}) \;=\; &\not{p}\Big\{\frac{\alpha}{\pi}\,\lambda\,\frac{1}{4}\,C_2(R)\,\Big[\Big(-\ln 4\pi+\gamma_E+\ln\frac{m^2}{\nu^2}\Big)\\
&-1-\frac{m^2}{p^2}+\left(1-\frac{m^4}{p^4}\right)\ln\left(1-\frac{p^2}{m^2}\right)\Big]+C_{2F}\Big\}\\
&+m\,\frac{\alpha}{\pi}\,\frac{1}{4}\,C_2(R)\,\Big\{-(3+\lambda)\Big(-\ln 4\pi+\gamma_E+\ln\frac{m^2}{\nu^2}\Big)\\
&+2(2+\lambda)-(3+\lambda)\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)-C_4\Big\}\,,
\end{aligned}
\tag{3.4.18}
$$

which is the desired, finite result for the dressed-quark self-energy.

It is not common to work explicitly with the counterterms. More often one uses Eq. (3.4.9) and the definition of the connected 2-point quark Green function (an obvious analogue of Eq. (3.1.15)):

$$
\begin{aligned}
iS_R^f(x,y;m,\lambda,\alpha) &= -\frac{\delta^2 Z_R[J^a_\mu,\xi,\bar\xi]}{\delta\xi^f(y)\delta\bar\xi^f(x)}\\
&= \langle 0|q^f(x)\,\bar q^f(y)|0\rangle = Z_{2F}^{-1}\,\langle 0|q_0^f(x)\,\bar q_0^f(y)|0\rangle
\end{aligned}
\tag{3.4.19}
$$

to write

$$
S_R(\not{p};m,\lambda,\alpha)=\lim_{\epsilon\to 0}\left\{Z_{2F}^{-1}\,S_0(\not{p};m_0,\lambda_0,\alpha_0;\epsilon)\right\}\,, \tag{3.4.20}
$$

where, in the r.h.s., m_0, λ_0, α_0 have to be substituted by their expressions in terms of the renormalised quantities and the limit taken order by order in α.

To illustrate this using our concrete example: Eq. (3.4.20) yields

$$\begin{aligned}\left\{1-\Sigma_{VR}(p^2;m,\lambda,\alpha)\right\}\not{p} - m\left\{1+\Sigma_{SR}(p^2;m,\lambda,\alpha)/m\right\} \\ = \lim_{\epsilon\to 0^-}\Big(Z_{2F}\Big\{\left[1-\Sigma_V(p^2;m_0,\lambda_0,\alpha_0)\right]\not{p} \\ -m_0\left[1+\Sigma_S(p^2;m_0,\lambda_0,\alpha_0)/m_0\right]\Big\}\Big) \qquad (3.4.21)\end{aligned}$$

and taking into account that $m = Z_4^{-1} Z_{2F} m_0$ then

$$1-\Sigma_{VR}(p^2;m,\lambda,\alpha) = \lim_{\epsilon\to 0^-} Z_{2F}\left\{1-\Sigma_V(p^2;m_0,\lambda_0,\alpha_0)\right\}, \qquad (3.4.22)$$

$$1+\Sigma_{SR}(p^2;m,\lambda,\alpha)/m = \lim_{\epsilon\to 0^-} Z_4\left\{1+\Sigma_S(p^2;m_0,\lambda_0,\alpha_0)/m_0\right\}. \qquad (3.4.23)$$

Now the renormalisation constants, Z_{2F}, Z_4, are chosen so as to exactly cancel the $1/\epsilon$ poles in the r.h.s. of Eqs. (3.4.22), (3.4.23). Using Eqs. (3.3.53), (3.3.54) then Eqs. (3.4.16), (3.4.17) are immediately reproduced.

Modified Minimal Subtraction

The modified minimal substraction scheme $\overline{\rm MS}$ is also often used in QCD. It takes advantage of the fact that the $1/\epsilon$ pole obtained using dimensional regularisation always appears in the combination

$$\frac{1}{\epsilon} - \ln 4\pi + \gamma_E \qquad (3.4.24)$$

by defining the renormalisation constants so as to eliminate this combination, in its entirety, from the Green functions. At one-loop order the renormalisation constants in the $\overline{\rm MS}$ scheme are trivially related to those in the MS scheme. At higher orders there are different ways of defining the scheme and the relation between the renormalisation constants is not so simple.

Momentum Subtraction

In the momentum subtraction scheme (μ-scheme) a given renormalised Green function, G_R, is obtained from its regularised counterpart, G, by subtracting from G its own value at some arbitrarily chosen momentum

scale. In QCD that scale is always chosen to be a Euclidean momentum: $p^2 = -\mu^2$. Returning to our example of the dressed-quark self-energy, in this scheme

$$\Sigma_{AR}(p^2;\mu^2) := \Sigma_A(p^2;\epsilon) - \Sigma_A(p^2;\epsilon)\,;\ A = V, S\,, \tag{3.4.25}$$

and so

$$\begin{aligned}
\Sigma^{(2)}_{VR}(p^2;\mu^2) &= \frac{\alpha(\mu)}{\pi}\,\lambda(\mu)\,\frac{1}{4}\,C_2(R)\,\left\{-m^2(\mu)\left(\frac{1}{p^2}+\frac{1}{\mu^2}\right)\right. \\
&\quad + \left(1-\frac{m^4(\mu)}{p^4}\right)\ln\left(1-\frac{p^2}{m(\mu)^2}\right) \\
&\quad \left. - \left(1-\frac{m^4(\mu)}{\mu^4}\right)\ln\left(1+\frac{\mu^2}{m^2(\mu)}\right)\right\}\,, \qquad (3.4.26)\\
\Sigma^{(2)}_{SR}(p^2;\mu^2) &= m(\mu)\,\frac{\alpha(\mu)}{\pi}\,\frac{1}{4}\,C_2(R)\,\left\{-[3+\lambda(\mu)]\right. \\
&\quad \times\left[\left(1-\frac{m^2(\mu)}{p^2}\right)\ln\left(1-\frac{p^2}{m^2(\mu)}\right)\right. \\
&\quad \left.\left. - \left(1+\frac{m^2(\mu)}{\mu^2}\right)\ln\left(1+\frac{\mu^2}{m^2(\mu)}\right)\right]\right\}, \qquad (3.4.27)
\end{aligned}$$

where the renormalised quantities depend on the point at which the renormalisation has been conducted. Clearly, from Eqs. (3.4.22), (3.4.23), the renormalisation constants in the μ-scheme are

$$\begin{aligned}
Z^{(2)}_{2F} &= 1 + \Sigma^{(2)}_V(p^2 = -\mu^2;\epsilon) \qquad (3.4.28)\\
Z^{(2)}_{4} &= 1 - \Sigma^{(2)}_S(p^2 = -\mu^2;\epsilon)/m(\mu)\,. \qquad (3.4.29)
\end{aligned}$$

It is apparent that in this scheme there is at least one point, the renormalisation mass-scale, μ, at which there are no higher order corrections to any of the Green functions: the corrections are all absorbed into the redefinitions of the coupling constant, masses and gauge parameter. This is valuable if the coefficients of the higher order corrections, calculated with the parameters defined through the momentum space subtraction, are small for then the procedure converges rapidly on a large momentum domain. In this sense the μ-scheme is easier to understand and more intuitive than the MS or $\overline{\rm MS}$ schemes. Another advantage is the manifest applicability of the "decoupling theorem," Refs. [SY73; AC75], which states that quark flavours whose masses are larger than the scale chosen for μ are irrelevant and hence can be neglected in all calculations.

This last feature, however, also emphasises that the renormalisation constants are flavour dependent and that can be a nuisance. Nevertheless, the μ-scheme is extremely useful in nonperturbative analyses of DSEs, especially since the flavour dependence of the renormalisation constants is minimal for light quarks when the Euclidean substraction point, μ, is chosen to be very large; i.e., much larger than their current-masses.

3.4.3 *Renormalised Gap Equation*

Equation (3.3.1) is the unrenormalised QCD gap equation, which can be rewritten as

$$-iS_0^{-1}(p) = -i(\not{p} - m_0) + g_0^2 \int \frac{d^4\ell}{(2\pi)^4}\, D_0^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a\gamma_\mu\, S_0(\ell)\, i\Gamma^a_{0\nu}(\ell,p)\,. \tag{3.4.30}$$

The renormalised equation can be derived directly from the generating functional defined using the renormalised Lagrangian density, Eq. (3.4.6), simply by repeating the steps described in Sec. 3.2. Alternatively, one can use Eqs. (3.4.9) to derive an array of relations similar to Eq. (3.4.19):

$$D_0^{\mu\nu}(k) = Z_{3YM}\, D_R^{\mu\nu}(k)\,, \quad \Gamma^a_{0\nu}(k,p) = Z_{1F}^{-1}\, \Gamma^a_{R\nu}(k,p)\,, \tag{3.4.31}$$

and others that are not needed here.[§] Now one can replace the unrenormalised couplings, masses and Green functions by their renormalised forms:

$$\begin{aligned} -iZ_2^{-1}\, S_R^{-1}(p) = {} & -i\not{p} + iZ_{2F}^{-1}\, Z_4\, m_R \\ & + Z_{1F}^2\, Z_{3YM}^{-1}\, Z_{2F}^{-2}\, g_R^2 \\ & \times \int \frac{d^4\ell}{(2\pi)^4}\, Z_{3YM}\, D_R^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a\gamma_\mu\, Z_{2F}\, S_R(\ell)\, Z_{1F}^{-1}\, i\Gamma^a_{R\nu}(\ell,p)\,, \end{aligned} \tag{3.4.32}$$

which simplifies to

$$\begin{aligned} -i\Sigma_R(p) \quad = \quad & i(Z_{2F}-1)\,\not{p} - i(Z_4-1)\, m_R \\ & - Z_{1F}\, g_R^2 \int \frac{d^4\ell}{(2\pi)^4}\, D_R^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a\gamma_\mu\, S_R(\ell)\, i\Gamma^a_{R\nu}(\ell,p) \\ =: \quad & i(Z_{2F}-1)\,\not{p} - i(Z_4-1)\, m_R - i\Sigma'(p)\,, \end{aligned} \tag{3.4.33}$$

[§]NB. It is a general feature that propagators are multiplied by the renormalisation constant and proper vertices by the inverse of renormalisation constants.

where $\Sigma'(p)$ is the regularised self-energy.

In the simplest application of the μ-scheme one would choose a large Euclidean mass-scale, μ^2, and define the renormalisation constants such that

$$\Sigma_R(\not{p}+\mu=0) = 0\,, \tag{3.4.34}$$

which entails

$$Z_{2F} = 1+\Sigma'_V(\not{p}+\mu=0)\,,\; Z_4 = 1-\Sigma'_S(\not{p}+\mu=0)/m_R(\mu)\,, \tag{3.4.35}$$

where I have used $\Sigma'(p) = \Sigma'(p)\,\not{p}+\Sigma'_S(p)$. (cf. Eqs. (3.4.28), (3.4.29).) This is simple to implement, even nonperturbatively, and is always appropriate in QCD because confinement ensures that dressed-quarks do not have a mass-shell.

On-shell renormalisation

If one is treating fermions that do have a mass-shell; e.g., electrons, then an on-shell renormalisation scheme may be more appropriate. One fixes the renormalisation constants such that

$$S_R^{-1}(\not{p})\big|_{\not{p}=m_R} = \not{p}-m_R\,, \tag{3.4.36}$$

which is interpreted as a constraint on the pole position and the residue at the pole; i.e., since

$$S^{-1}(\not{p}) = [\not{p}-m_R-\Sigma_R(\not{p})]\big|_{\not{p}=m_R} - [\not{p}-m_R]\left[\frac{d}{d\not{p}}\Sigma_R(\not{p})\right]\Big|_{\not{p}=m_R} + \ldots \tag{3.4.37}$$

then Eq. (3.4.36) entails

$$\Sigma_R(\not{p})\big|_{\not{p}=m_R} = 0 \;\text{ and }\; \frac{d}{d\not{p}}\Sigma_R(\not{p})\Big|_{\not{p}=m_R} = 0\,. \tag{3.4.38}$$

The second of these equations gives

$$Z_{2F} = 1+\Sigma'_V(m_R^2) + 2\,m_R^2\,\frac{d}{dp^2}\,\Sigma'_V(p^2)\big|_{p^2=m_R^2} + 2\,m_R\,\frac{d}{dp^2}\Sigma'_S(p^2)\Big|_{p^2=m_R^2} \tag{3.4.39}$$

and the first:

$$Z_4 = Z_{2F} - \Sigma'_V(m_R^2) - \Sigma'_S(m_R^2)\,. \tag{3.4.40}$$

3.5 Exercises

(1) Verify Eq. (3.1.10).
(2) Verify Eq. (3.1.20).
(3) Verify Eq. (3.1.29).
(4) Verify Eq. (3.2.4).
(5) Verify Eq. (3.3.15).
(6) Verify Eq. (3.3.21).
(7) Verify Eq. (3.3.27).
(8) Verify Eqs. (3.3.42) and (3.3.43).
(9) Verify Eqs. (3.3.53) and (3.3.54).
(10) Using Eqs. (3.3.53), (3.3.54) derive Eqs. (3.4.16), (3.4.17).
(11) Verify Eqs. (3.4.28), (3.4.29).
(12) Verify Eq. (3.4.33).
(13) Verify Eq. (3.4.35).
(14) Beginning with Eq. (3.4.36), derive Eqs. (3.4.39), (3.4.40). NB. $\not{p}\not{p} = p^2$ and hence $\frac{d}{d\not{p}} f(p^2) = \frac{d}{d\not{p}} f(\not{p}\not{p}) = 2\,\not{p}\,\frac{d}{dp^2} f(p^2)$.

Chapter 4
Dynamical Chiral Symmetry Breaking

This phenomenon profoundly affects the character of the hadron spectrum. However, it is intrinsically nonperturbative and therefore its understanding is best sought via a Euclidean formulation of quantum field theory.

4.1 Euclidean Metric

Free Scalar Field

We will formally describe the construction of the Euclidean counterpart to a given Minkowski space field theory using the free scalar field as an example. The action that appears in the generating functional for a free scalar quantum field theory is given in Eq. (2.2.20):

$$\begin{aligned}&\int d^4x\,\frac{1}{2}[\partial_\mu\phi(x)\,\partial^\mu\phi(x) - m^2\phi^2(x) + i\eta^+\,\phi^2(x)]\\ &\quad= \int_{-\infty}^{\infty} dt \int d^3x\;\phi(x)\,\frac{1}{2}[-\partial^2 - m^2 + i\eta^+]\,\phi(x)\,. \qquad (4.1.1)\end{aligned}$$

At the end of Sec. 2.1 we remarked upon the role of $i\eta^+$ here: it was introduced to provide a damping factor in the generating functional. The alternative proposed was to change variables and introduce a Euclidean time: $t \to -it^E$. As usual

$$\int_{-\infty}^{\infty} dt\, f(t) = \int_{-\infty}^{\infty} d(-it^E)\, f(-it^E) = -i\int_{-\infty}^{\infty} dt^E\, f(-it^E) \qquad (4.1.2)$$

if $f(t)$ vanishes on the arc at infinity in the second and fourth quadrants of the complex-t plane, and is analytic therein.

In order to apply this to Eq. (4.1.1), we use Eq. (1.2.10) and observe that

$$\partial^2 = \frac{\partial}{\partial t}\frac{\partial}{\partial t} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} \stackrel{M\to E}{\to} -\frac{\partial}{\partial t^E}\frac{\partial}{\partial t^E} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^E_\mu}\frac{\partial}{\partial x^E_\mu} =: -\partial^2_E\,, \tag{4.1.3}$$

where $(x^E_\mu) = (\vec{x}, x_4 := -it)$, and hence

$$\begin{aligned}\int_{-\infty}^{\infty} dt \int d^3x\ \phi(x)\left[-\partial^2 - m^2 + i\eta^+\right]\phi(x) \\ = \ i\int_{-\infty}^{\infty} dt^E \int d^3x\,\phi(x^E)\left[-\partial^2_E + m^2 - i\eta^+\right]\phi(x^E)\,.\end{aligned} \tag{4.1.4}$$

The generating functional for a free scalar field theory is given in Eq. (2.2.18):

$$W_0[J] = \int [\mathcal{D}\phi(x)]\ \mathrm{e}^{i\int d^4x\left\{\phi(x)\frac{1}{2}\left[-\partial^2 - m^2 + i\eta^+\right]\phi(x) + J(x)\phi(x)\right\}}, \tag{4.1.5}$$

and using the Wick rotation formulae just established we are led to a natural definition of the Euclidean generating functional for a free scalar field theory:

$$W^E_0[J] = \int [\mathcal{D}\phi(x^E)]\ \mathrm{e}^{-\int d^4x^E\left\{\phi(x^E)\frac{1}{2}\left[-\partial^2_E + m^2 + i\eta^+\right]\phi(x^E) - J(x^E)\phi(x^E)\right\}}. \tag{4.1.6}$$

Let's focus on the action in this equation:

$$A_E = \int d^4x^E\,\phi(x^E)\,\frac{1}{2}\left[-\partial^2_E + m^2 - i\eta^+\right]\phi(x^E)\,. \tag{4.1.7}$$

Its complex conjugate is

$$\begin{aligned} A^*_E &= \int d^4x^E\,\left\{\phi(x^E)\,\frac{1}{2}\left[-\partial^2_E + m^2 - i\eta^+\right]\phi(x^E)\right\}^* \\ &= \int d^4x^E\,\phi(x^E)\,\frac{1}{2}\left[-\partial^2_E + m^2 + i\eta^+\right]\phi(x^E) \end{aligned} \tag{4.1.8}$$

because the mass is real and ∂^2_E is an Hermitian operator. (NB. Naturally, the neutral scalar field is also real.) Consequently, in the absence of the factor $i\eta^+$, the action is real. It is also positive definite because all the eigenvalues of the operator $(-\partial^2_E + m^2)$ are positive. Hence the source-independent part of the integrand in Eq. (4.1.6) truly defines a Gaussian

probability measure and this establishes that the convergence factor $i\eta^+$ is unnecessary in the Euclidean formulation.

We observe that Eqs. (4.1.5) and (4.1.6) are rigorously related via a Wick rotation at every order in perturbation theory. However, that relationship can be destroyed by nonperturbative effects. Furthermore, the procedure we have outlined can also be followed to arrive at the Euclidean space generating functional for interacting scalar field theories. In these cases, too, the above caveat applies.

Free Fermion Field

Analogous steps can be used to motivate the form of the Euclidean generating functional for fermions and since there are some additional features we outline the procedure. The Lagrangian density for free Dirac fields is given in Eq. (2.3.46):

$$L_0^{\psi}(x) = \int_{-\infty}^{\infty} dt \int d^3x\, \bar{\psi}(x) \left(i\partial\!\!\!/ - m + i\eta^+ \right) \psi(x)\,. \tag{4.1.9}$$

Clearly, in order to implement Eq. (4.1.2) we must determine the effect of a Wick rotation on $i\partial\!\!\!/$, since that is part of the the integrand in Eq. (4.1.9):

$$\begin{aligned} i\partial\!\!\!/ = i\gamma^0 \frac{\partial}{\partial t} + i\gamma^i \frac{\partial}{\partial x^i} \quad &\overset{M\to E}{\to} \quad i\gamma^0 \frac{\partial}{\partial(-it^E)} + i\gamma^i \frac{\partial}{\partial x^i} \\ &= \quad -\gamma^0 \frac{\partial}{\partial t^E} - (-i\gamma^i) \frac{\partial}{\partial x^i} \qquad (4.1.10) \\ &=: \quad -\gamma_\mu^E \frac{\partial}{\partial x_\mu^E}\,, \qquad (4.1.11) \end{aligned}$$

where the Euclidean Dirac matrices are

$$\gamma_4^E = \gamma^0\,;\ \gamma_i^E = -i\gamma^i\,,\ i = 1,2,3. \tag{4.1.12}$$

These matrices are Hermitian and satisfy the algebra

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\,\delta_{\mu\nu}\,;\ \mu,\nu = 1,\ldots,4, \tag{4.1.13}$$

where $\delta_{\mu\nu}$ is the four-dimensional Kronecker delta. Henceforth we will adopt the notation

$$a^E \cdot b^E = \sum_{\mu=1}^{4} a_\mu^E b_\nu^E\,. \tag{4.1.14}$$

Using these results and assuming that the integrand in Eq. (4.1.9) is analytic where necessary, one arrives formally at

$$\begin{aligned}\int_{-\infty}^{\infty} dt \int d^3x\, \bar{\psi}(x) \left(i\partial\!\!\!/ - m + i\eta^+\right) \psi(x) \\ = \quad i\int_{-\infty}^{\infty} dt^E \int d^3x\, \bar{\psi}(x^E) \left(\gamma^E \cdot \partial^E + m - i\eta^+\right) \psi(x^E)\,.\end{aligned} \tag{4.1.15}$$

Hence in connection with the generating functional of Eq. (2.3.47):

$$\begin{aligned}W_0[\bar{\xi},\xi] = \int [\mathcal{D}\bar{\psi}(x)][\mathcal{D}\psi(x)] \\ \times \exp\left\{ i\int d^4x \left[\bar{\psi}(x)\left(i\partial\!\!\!/ - m + i\eta^+\right)\psi(x) + \bar{\psi}(x)\xi(x) + \bar{\xi}(x)\psi(x)\right]\right\},\end{aligned} \tag{4.1.16}$$

we are led to the following definition of the Euclidean generating functional for a free fermion field theory:

$$\begin{aligned}W_0^E[\bar{\xi},\xi] := \int [\mathcal{D}\bar{\psi}(x^E)][\mathcal{D}\psi(x^E)] \\ \times \exp\left\{ -\int d^4x^E \left[\bar{\psi}(x^E)\left(\gamma\cdot\partial + m\right)\psi(x^E)\right.\right. \\ \left.\left. -\bar{\psi}(x^E)\xi(x^E) - \bar{\xi}(x^E)\psi(x^E)\right]\right\},\end{aligned} \tag{4.1.17}$$

where we have dropped the "$+i\eta^+$" convergence factor. As in the scalar case: Eqs. (4.1.16) and (4.1.17) are simply related via a Wick rotation at every order in perturbation theory but nonperturbative effects can destroy that connection; and the same procedure can be used to arrive at the Euclidean generating functional for interacting field theories with fermions.

Euclidean Formulation as Definitive

This discussion makes clear that working in Euclidean space is more than simply a pragmatic artifice. Indeed, it is possible to view the Euclidean formulation of a quantum field theory as definitive (see, for example, Refs. [SY69; SW80; GJ81; SE82]). That decision is crucial when a consideration of nonperturbative effects becomes important. In addition, the discrete lattice formulation in Euclidean space has allowed some progress to be made in attempting to answer existence questions for interacting gauge

field theories [SE82]. As we have repeatedly observed, a simple lattice formulation is not possible in Minkowski space because of the nature of the measure.

In adopting the view that Euclidean generating functionals are definitive one relies on a few results from axiomatic field theory, as we now explain.

The moments of the Euclidean measure defined by an interacting quantum field theory are the Schwinger functions:

$$\mathcal{S}^n(x^{E,1},\ldots,x^{E,n}) \tag{4.1.18}$$

which can be obtained as usual via functional differentiation of the analogue of Eq. (4.1.17) and subsequently setting the sources to zero. The Schwinger functions are sometimes called Euclidean space Green functions.

Given a measure and given that it satisfies certain conditions (i.e., the Wightman and Haag-Kastler axioms), then it can be shown that the Wightman functions, $\mathcal{W}^n(x_1,\ldots,x_n)$, can be obtained from the Schwinger functions by analytic continuation in each of the time coordinates:

$$\mathcal{W}^n(x_1,\ldots,x_n) = \lim_{x_4^i\to 0} \mathcal{S}^n([\vec{x}^1, x_4^1 + ix_1^0],\ldots,[\vec{x}^n, x_4^n + ix_n^0])\,, \tag{4.1.19}$$

with $x_1^0 < x_2^0 < \ldots < x_n^0$. These Wightman functions are simply the vacuum expectation values of products of field operators from which the Green functions (i.e., the Minkowski space propagators) are obtained through the inclusion of step- [θ-] functions in order to obtain the appropriate time ordering. (This is described in some detail in Refs. [SW80; GJ81; SE82].) Thus the Schwinger functions contain all of the information necessary to calculate physical observables.

This notion is used directly in obtaining masses and charge radii in lattice simulations of QCD, and it can also be employed in the DSE approach because the Euclidean space DSEs can all be derived from the appropriate Euclidean generating functional using the methods of Sec. 2.5 and the solutions of these equations are the Schwinger functions.

All this provides a good reason to employ a Euclidean formulation. Another is a desire to maintain a direct and rigorous connection with perturbation theory because the renormalisation group equations for QCD and their solutions are best understood in Euclidean space [GR75].

Minkowski ↔ Euclidean Transcription Formulae

To make clear our conventions (henceforth we will omit the superscript E

used to denote Euclidean four-vectors): for 4-vectors a, b:

$$a \cdot b := a_\mu \, b_\nu \, \delta_{\mu\nu} := \sum_{i=1}^{4} a_i \, b_i \,, \tag{4.1.20}$$

so that a spacelike vector, Q_μ, has $Q^2 > 0$; the Dirac matrices are Hermitian and defined by the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2 \, \delta_{\mu\nu} \,; \tag{4.1.21}$$

and we use

$$\gamma_5 := -\gamma_1\gamma_2\gamma_3\gamma_4 \tag{4.1.22}$$

so that

$$\mathrm{tr}\,[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = -4 \, \varepsilon_{\mu\nu\rho\sigma} \,, \; \varepsilon_{1234} = 1 \,. \tag{4.1.23}$$

The Dirac-like representation of these matrices is:

$$\vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\tau} \\ i\vec{\tau} & 0 \end{pmatrix}, \; \gamma_4 = \begin{pmatrix} \tau^0 & 0 \\ 0 & -\tau^0 \end{pmatrix}, \tag{4.1.24}$$

where the 2×2 Pauli matrices are:

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1.25}$$

Using these conventions the [unrenormalised] Euclidean QCD action is

$$S[B, q, \bar{q}] = \int d^4x \, \left[\frac{1}{4} \, F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\lambda} \, \partial \cdot B^a \, \partial \cdot B^a \right. \\ \left. + \sum_{f=1}^{N_f} \, \bar{q}_f \left(\gamma \cdot \partial + m_f + ig \, \frac{1}{2} \lambda^a \, \gamma \cdot B^a \right) q_f \right], \tag{4.1.26}$$

where $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g f^{abc} B^b_\mu B^c_\nu$. The generating functional follows:

$$W[J, \xi, \bar{\xi}] = \int \, d\mu(\bar{q}, q, B, \bar{\omega}, \omega) \, \exp \int d^4x \, \left[\bar{q} \, \xi + \bar{\xi} \, q + J^a_\mu \, A^a_\mu \right], \tag{4.1.27}$$

with sources: $\bar{\eta}$, η, J, and a functional integral measure

$$d\mu(\bar{q}, q, B, \bar{\omega}, \omega) := \prod_x \prod_\phi \mathcal{D}\bar{q}_\phi(x) \mathcal{D}q_\phi(x) \prod_a \mathcal{D}\bar{\omega}^a(x) \mathcal{D}\omega^a(x)$$

$$\times \prod_{\mu} \mathcal{D}B^a_\mu(x) \exp(-S[B,q,\bar{q}] - S^g[B,\omega,\bar{\omega}]) \,, \quad (4.1.28)$$

where ϕ represents both the flavour and colour index of the quark field, and $\bar{\omega}$ and ω are the scalar, Grassmann [ghost] fields. The normalisation

$$W[\bar{\eta}=0,\eta=0,J=0]=1 \quad (4.1.29)$$

is implicit in the measure. As we saw in Sec. 2.4.1, the ghosts only couple directly to the gauge field:

$$S_g[B,\omega,\bar{\omega}] = \int d^4x \, \left[-\partial_\mu\bar{\omega}^a \, \partial_\mu\omega^a - g f^{abc} \, \partial_\mu\bar{\omega}^a \, \omega^b \, B^c_\mu\right] \,, \quad (4.1.30)$$

and restore unitarity in the subspace of transverse [physical] gauge fields. We note that, practically, the normalisation means ghost fields are unnecessary in the calculation of gauge invariant observables using lattice-regularised QCD because the gauge-orbit volume-divergence in the generating functional, which is associated with the uncountable infinity of gauge-equivalent gluon field configurations in the continuum, is rendered finite by the simple expedient of only summing over a finite number of configurations.

As we have observed, it is possible to derive every equation introduced above assuming certain analytic properties of the integrands. However, the derivations can be sidestepped using the following *transcription rules*:

Configuration Space	Momentum Space
(1) $\int^M d^4x^M \to -i \int^E d^4x^E$	(1) $\int^M d^4k^M \to i \int^E d^4k^E$
(2) $\not{\partial} \to i\gamma^E \cdot \partial^E$	(2) $\not{k} \to -i\gamma^E \cdot k^E$
(3) $\not{A} \to -i\gamma^E \cdot A^E$	(3) $\not{A} \to -i\gamma^E \cdot A^E$
(4) $A_\mu B^\mu \to -A^E \cdot B^E$	(4) $k_\mu q^\mu \to -k^E \cdot q^E$
(5) $x^\mu \partial_\mu \to x^E \cdot \partial^E$	(5) $k_\mu x^\mu \to -k^E \cdot x^E$

These rules are valid in perturbation theory; i.e., the correct Minkowski space integral for a given diagram will be obtained by applying these rules to the Euclidean integral: they take account of the change of variables and rotation of the contour. However, for diagrams that represent DSEs which involve dressed n-point functions, whose analytic structure is not known *as priori*, the Minkowski space equation obtained using this prescription will have the right appearance but it's solutions may bear no relation to the

analytic continuation of the solution of the Euclidean equation. Any such differences will be nonperturbative in origin.

4.2 Chiral Symmetry

Gauge theories with massless fermions have a chiral symmetry. Its effect can be visualised by considering the particle's helicity: $\lambda \propto J \cdot p$, the projection of the fermion's spin onto its direction of motion. λ is a Poincaré invariant spin observable that takes a value of ± 1. The chirality operator can be realised on the massless Dirac spinor basis via the 4×4-matrix, γ_5, and the eigenvalue equations

$$\gamma_5 \, q(x) = \pm \, q(x) \tag{4.2.1}$$

have solutions

$$q_R(x) := q_{\lambda=+1}(x) \;\; = \;\; \frac{1}{2} \left(\mathbf{1} + \gamma_5 \right) \, q(x) =: P_R \, q(x) \,, \tag{4.2.2}$$

$$q_L(x) := q_{\lambda=-1}(x) \;\; = \;\; \frac{1}{2} \left(\mathbf{1} - \gamma_5 \right) \, q(x) =: P_L \, q(x) \,, \tag{4.2.3}$$

for an arbitrary spinor, $q(x)$. All spinors can be decomposed as $q(x) = q_R(x) + q_L(x)$, and for massless fermions $q_R(x)$ and $q_L(x)$ do not mix under proper Lorentz transformations. However, the parity or space-reflection operator, which is an improper Lorentz transformation and is realised on fermions via

$$q(x) \to q'(x') = \gamma_4 \, q(x) \tag{4.2.4}$$

(we have neglected un unimportant phase), flips the helicity of massless particles. That is easily verified and corresponds to intuition: under space-reflection angular momentum, which is a three-dimensional pseudovector, doesn't change sign but the momentum three-vector does.

A general chiral transformation effects a rotation of the fermion field

$$q(x) \to q(x)' = \mathrm{e}^{i \gamma_5 \theta} \, q(x) \,, \tag{4.2.5}$$

and this is expressed on the helicity eigenstates via

$$\mathrm{e}^{i \gamma_5 \theta} \, q_R(x) = \mathrm{e}^{i \theta} \, q_R(x) \,, \;\; \mathrm{e}^{i \gamma_5 \theta} \, q_L(x) = \mathrm{e}^{-i \theta} \, q_L(x) \,. \tag{4.2.6}$$

It follows from these equations that

$$\bar{q}(x) \to \bar{q}(x)' = \bar{q}(x)\, \mathrm{e}^{i\gamma_5\theta} \tag{4.2.7}$$

and

$$\bar{q}_R(x) \to \bar{q}_R(x)\, \mathrm{e}^{-i\theta}\,, \quad \bar{q}_L(x) \to \bar{q}_L(x)\mathrm{e}^{i\theta}\, q_L(x)\,. \tag{4.2.8}$$

The simplest composite operator that can create a pseudoscalar field is

$$\pi(x) := \bar{q}(x) i\gamma_5 q(x) = i\bar{q}_L(x)\, q_R(x) - i\bar{q}_R(x)\, q_L(x)\,, \tag{4.2.9}$$

which is obviously odd under the parity transformation since that simply flips the helicity of each field:

$$\pi(x) \overset{\text{Parity}}{\to} \pi'(x') = i\bar{q}_R(x)\, q_L(x) - i\bar{q}_L(x)\, q_R(x) = -\pi(x)\,. \tag{4.2.10}$$

Now, under a general chiral rotation

$$\pi(x) \to \pi(x)' = \mathrm{e}^{2i\theta} i\bar{q}_L(x)\, q_R(x) - \mathrm{e}^{-2i\theta} i\bar{q}_R(x)\, q_L(x) \tag{4.2.11}$$

and hence with $\theta = -\pi/4$ we have

$$\pi(x)' = \bar{q}_L(x)\, q_R(x) + \bar{q}_R(x)\, q_L(x) = \bar{q}(x) q(x) =: \sigma(x)\,, \tag{4.2.12}$$

which is obviously even under the parity transformation. This means that a chiral rotation can change an odd parity state into one with even parity: $J^{P=-} \to J^{P=+}$, where J is the states' total angular momentum.

In a theory with chiral symmetry the Hamiltonian (or mass operator) must commute with all operators that express chiral transformations. (This is merely another way of expressing what it means for a theory to possess the symmetry.) Hence any two states related by a chiral transformation must be degenerate, from which it follows that the states created by $\pi(x)$ are degenerate with those created by $\sigma(x)$; i.e., the spectrum of a theory with chiral symmetry should exhibit degenerate parity multiplets.

Turning now to QCD, we observe that for many reasons, the masses of the u- and d-quarks are expected to be very small; i.e., $m_u \sim m_d \ll \Lambda_{\mathrm{QCD}}$, where $\Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV}$ is a mass-scale characterising the theory. Therefore chiral symmetry should only be weakly broken and hence, following the above argument, the strong interaction spectrum should exhibit nearly degenerate parity partners. The experimental comparison is presented in

Eq. (4.2.13):

$$\begin{array}{c|c|c} N(\frac{1}{2}^{P=+},938) & \pi(0^{P=-},140) & \rho(1^{P=-},770) \\ N(\frac{1}{2}^{P=-},1535) & a_0(0^{P=+},980) & a_1(1^{P=+},1260) \end{array} \quad . \tag{4.2.13}$$

Clearly the expectation is very badly violated, with the splitting much too large to be described by the small current-quark masses. What is wrong?

Chiral symmetry can be related to properties of the quark propagator, $S(p)$. For a free quark (remember, we are now using Euclidean conventions)

$$S_0(p) = \frac{m - i\gamma\cdot p}{m^2 + p^2} \tag{4.2.14}$$

and as a matrix

$$S_0(p) \quad \to \quad \mathrm{e}^{i\gamma_5\theta} S_0(p) \mathrm{e}^{i\gamma_5\theta} = \frac{-i\gamma\cdot p}{p^2+m^2} + \mathrm{e}^{2i\gamma_5\theta}\frac{m}{p^2+m^2} \tag{4.2.15}$$

under a chiral transformation. As anticipated, for $m = 0$, $S_0(p) \to S_0(p)$; i.e., the symmetry breaking term is proportional to the current-quark mass and it can be measured by the "quark condensate"

$$-\langle\bar{q}q\rangle := \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\,[S(p)] \propto \int \frac{d^4p}{(2\pi)^4}\,\frac{m}{p^2+m^2}\,, \tag{4.2.16}$$

which is the "Cooper-pair" density in QCD. For a free quark the condensate vanishes if $m = 0$ but what is the effect of interactions?

As we have seen, interactions dress the quark propagator so that it takes the form

$$S(p) := \frac{1}{i\gamma\cdot p + \Sigma(p)} = \frac{-i\gamma\cdot p A(p^2) + B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}\,, \tag{4.2.17}$$

where $\Sigma(p)$ is the self energy, expressed in terms of the scalar functions: A and B, which are p^2-dependent because the interaction is momentum-dependent. On the valid domain; i.e., for weak-coupling, they can be calculated in perturbation theory and at one-loop order, Eq. (3.4.27) with $p^2 \to -(p^E)^2 \gg \mu^2$, $m^2(\mu)$,

$$B(p^2) = m\left(1 - \frac{\alpha_S}{\pi}\ln\left[\frac{p^2}{m^2}\right]\right)\,, \tag{4.2.18}$$

which is $\propto m$. This result persists: at every order in perturbation theory every mass-like correction to $S(p)$ is $\propto m$ so that m is apparently the *only* source of chiral symmetry breaking and $\langle\bar{q}q\rangle \propto m \to 0$ as $m \to 0$. The

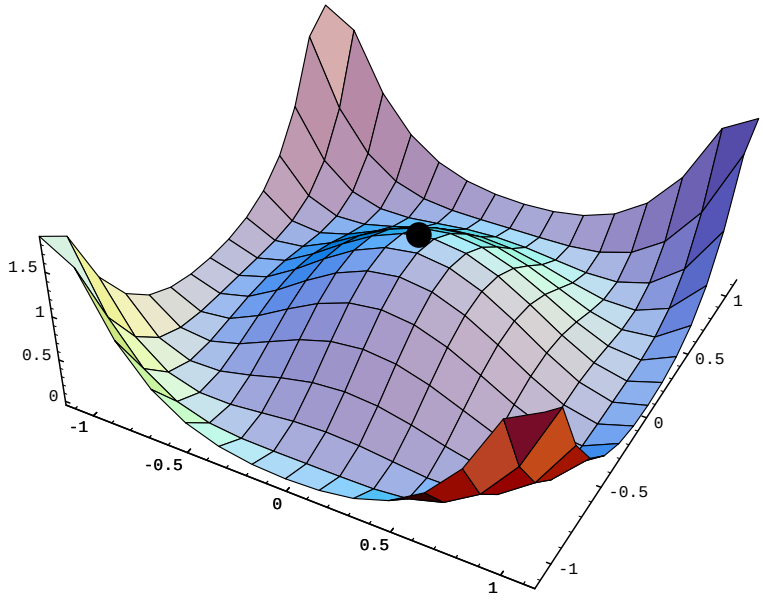

Fig. 4.1 A rotationally invariant but unstable extremum of the Hamiltonian obtained with the potential $V(\sigma,\pi) = (\sigma^2 + \pi^2 - 1)^2$.

current-quark masses are the only explicit chiral symmetry breaking terms in QCD.

However, symmetries can be "dynamically" broken. Consider a point-particle in a rotationally invariant potential $V(\sigma,\pi) = (\sigma^2+\pi^2-1)^2$, where (σ,π) are the particle's coordinates. In the state depicted in Fig. 4.1, the particle is stationary at an extremum of the action that is rotationally invariant but unstable. In the ground state of the system, the particle is stationary at any point (σ,π) in the trough of the potential, for which $\sigma^2 + \pi^2 = 1$. There are an uncountable infinity of such vacua, $|\theta\rangle$, which are related one to another by rotations in the (σ,π)-plane. The vacua are degenerate but not rotationally invariant and hence, in general, $\langle\theta|\sigma|\theta\rangle \neq 0$. In this case the rotational invariance of the Hamiltonian is not exhibited in any single ground state: the symmetry is dynamically broken with interactions being responsible for $\langle\theta|\sigma|\theta\rangle \neq 0$.

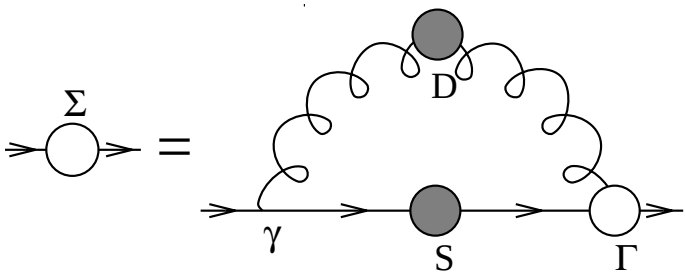

Fig. 4.2 DSE for the dressed-quark self-energy. The kernel of this equation is constructed from the dressed-gluon propagator (D - spring) and the dressed-quark-gluon vertex (Γ - open circle). One of the vertices is bare (labelled by γ) as required to avoid over-counting.

4.3 Mass Where There Was None

The analogue in QCD is $\langle\bar{q}q\rangle \neq 0$ when $m = 0$. At any finite order in perturbation theory that is impossible. However, using the DSE for the quark self energy (the QCD "gap equation"):

$$\begin{aligned} i\gamma\cdot p\, A(p^2) + B(p^2) &= Z_2\, i\gamma\cdot p + Z_4\, m \\ &+ Z_1 \int^\Lambda \frac{d^4\ell}{(2\pi)^4}\, g^2\, D_{\mu\nu}(p-\ell)\, \gamma_\mu \frac{\lambda^a}{2} \frac{1}{i\gamma\cdot\ell A(\ell^2) + B(\ell^2)} \Gamma^a_\nu(\ell,p)\,, \end{aligned} \tag{4.3.1}$$

which is depicted in Fig. 4.2 and is the Euclidean space analogue of Eq. (3.4.33), it is possible to sum infinitely many contributions. That allows one to expose effects in QCD which are inaccessible in perturbation theory.*

The quark DSE is a nonlinear integral equation for A and B, and it is the nonlinearity that makes possible a generation of nonperturbative effects.

*NB. In Eq. (4.3.1), m is the Λ-dependent current-quark bare mass and $\int^\Lambda$ represents mnemonically a *translationally-invariant* regularisation of the integral, with Λ the regularisation mass-scale. The final stage of any calculation is to remove the regularisation by taking the limit $\Lambda \to \infty$. The quark-gluon-vertex and quark wave function renormalisation constants, $Z_1(\zeta^2,\Lambda^2)$ and $Z_2(\zeta^2,\Lambda^2)$, depend on the renormalisation point, ζ, and the regularisation mass-scale, as does the mass renormalisation constant $Z_m(\zeta^2,\Lambda^2) := Z_2(\zeta^2,\Lambda^2)^{-1} Z_4(\zeta^2,\Lambda^2)$. These are features of renormalisation that follow from the discussion in Sec. 3.4.

The kernel of the equation is composed of the dressed-gluon propagator:

$$g^2 D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{\mathcal{G}(k^2)}{k^2}\,,\ \mathcal{G}(k^2) := \frac{g^2}{[1+\Pi(k^2)]}\,, \tag{4.3.2}$$

where $\Pi(k^2)$ is the vacuum polarisation, which contains all the dynamical information about gluon propagation, and the dressed-quark-gluon vertex: $\Gamma^a_\mu(k,p)$. The bare (undressed) vertex is

$$\Gamma^a_\mu(k,p)_{\rm bare} = \gamma_\mu \frac{\lambda^a}{2}\,. \tag{4.3.3}$$

Once $D_{\mu\nu}$ and Γ^a_μ are known, Eq. (4.3.1) is straightforward to solve by iteration. One chooses an initial seed for the solution functions: ${}_0A$ and ${}_0B$, and evaluates the integral on the r.h.s. The bare propagator values: ${}_0A = 1$ and ${}_0B = m$ are often adequate. This first iteration yields new functions: ${}_1A$ and ${}_1B$, which are reintroduced on the r.h.s. to yield ${}_2A$ and ${}_2B$, etc. The procedure is repeated until ${}_nA = {}_{n+1}A$ and ${}_nB = {}_{n+1}B$ to the desired accuracy.

It is now easy to illustrate DCSB, and we will use three simple examples.

4.3.1 *Nambu–Jona-Lasinio Model*

The Nambu–Jona-Lasinio model [NJ61] has been popularised as a model of low-energy QCD. The commonly used gap equation is obtained from Eq. (4.3.1) via the substitution

$$g^2 D_{\mu\nu}(p-\ell) \to \delta_{\mu\nu} \frac{1}{m_G^2}\,\theta(\Lambda^2-\ell^2) \tag{4.3.4}$$

in combination with Eq. (4.3.3). The step-function in Eq. (4.3.4) provides a momentum-space cutoff. That is necessary to define the model, which is not renormalisable and hence can be regularised but not renormalised. The cutoff, Λ, therefore persists as a model-parameter: it is an external mass-scale that cannot be eliminated.

The gap equation is

$$\begin{aligned} & i\gamma\cdot p\, A(p^2) + B(p^2) \\ & \quad = \quad i\gamma\cdot p + m + \frac{4}{3}\,\frac{1}{m_G^2} \int \frac{d^4\ell}{(2\pi)^4}\,\theta(\Lambda^2-\ell^2)\,\gamma_\mu\, \frac{-i\gamma\cdot\ell A(\ell^2)+B(\ell^2)}{\ell^2 A^2(\ell^2)+B^2(\ell^2)}\,\gamma_\mu\,, \end{aligned} \tag{4.3.5}$$

where we have set the renormalisation constants equal to one in order to complete the defintion of the model. Multiplying Eq. (4.3.5) by $(-i\gamma\cdot p)$ and tracing over Dirac indices, one obtains

$$p^2\,A(p^2) = p^2 + \frac{8}{3}\,\frac{1}{m_G^2}\int\frac{d^4\ell}{(2\pi)^4}\,\theta(\Lambda^2-\ell^2)\,p\cdot\ell\,\frac{A(\ell^2)}{\ell^2A^2(\ell^2)+B^2(\ell^2)}\,, \quad (4.3.6)$$

from which it is immediately apparent that

$$A(p^2)\equiv 1\,. \quad (4.3.7)$$

This property owes itself to the the fact that the NJL model is defined by a four-fermion contact interaction in configuration space, which entails the momentum-independence of the interaction in momentum space.

Simply tracing over Dirac indices and using Eq. (4.3.7), one obtains

$$B(p^2) = m + \frac{16}{3}\,\frac{1}{m_G^2}\int\frac{d^4\ell}{(2\pi)^4}\,\theta(\Lambda^2-\ell^2)\,\frac{B(\ell^2)}{\ell^2+B^2(\ell^2)}\,, \quad (4.3.8)$$

from which it is plain that $B(p^2)=\text{constant}=M$ is the only solution. This, too, is a result of the momentum-independence of the model's interaction. Evaluating the angular integrals, Eq. (4.3.8) becomes

$$\begin{aligned} M &= m + \frac{1}{3\pi^2}\,\frac{1}{m_G^2}\int_0^{\Lambda^2} dx\,x\,\frac{M}{x+M^2} && (4.3.9)\\ &= m + M\,\frac{1}{3\pi^2}\,\frac{1}{m_G^2}\,\mathcal{C}(M^2,\Lambda^2)\,, && (4.3.10)\\ \mathcal{C}(M^2,\Lambda^2) &= \Lambda^2 - M^2\ln\left[1+\Lambda^2/M^2\right]\,. && (4.3.11)\end{aligned}$$

The cutoff, Λ, defines the model's mass-scale and henceforth we set it equal to one so that all other dimensioned quantities are given in units of this scale, in which case the gap equation can be written

$$M = m + M\,\frac{1}{3\pi^2}\,\frac{1}{m_G^2}\,\mathcal{C}(M^2,1)\,. \quad (4.3.12)$$

Irrespective of the value of m_G, this equation always admits a solution $M\neq 0$ when the current-quark mass $m\neq 0$.

Consider now the chiral-limit, $m=0$, wherein the gap equation is

$$M = M\,\frac{1}{3\pi^2}\,\frac{1}{m_G^2}\,\mathcal{C}(M^2,1)\,. \quad (4.3.13)$$

This equation admits a solution $M \equiv 0$, which corresponds to the perturbative case considered above: when the bare mass of the fermion is zero in the beginning then no mass is generated via interactions. It follows from Eq. (4.2.16) that the condensate is also zero. This situation can be described as that of a theory without a mass gap: the negative energy Dirac sea is populated all the way up to $E = 0$.

Suppose however that $M \neq 0$ in Eq. (4.3.13), then the equation becomes

$$1 = \frac{1}{3\pi^2}\,\frac{1}{m_G^2}\,\mathcal{C}(M^2, 1)\,. \tag{4.3.14}$$

It is easy to see that $\mathcal{C}(M^2, 1)$ is a monotonically decreasing function of M with a maximum value at $M = 0$: $\mathcal{C}(0, 1) = 1$. Consequently Eq. (4.3.14) has a $M \neq 0$ solution if, and only if,

$$\frac{1}{3\pi^2}\,\frac{1}{m_G^2} > 1\,; \tag{4.3.15}$$

i.e., if and only if

$$m_G^2 < \frac{\Lambda^2}{3\pi^2} \simeq (0.2\,GeV)^2 \tag{4.3.16}$$

for a typical value of $\Lambda \sim 1\,\mathrm{GeV}$. Thus, even when the bare mass is zero, the NJL model admits a dynamically generated mass for the fermion when the coupling exceeds a given minimum value, which is called the critical coupling: the chiral symmetry is dynamically broken! (Here the critical coupling is expressed in a maximum value of the dynamical gluon mass, m_G^c: chiral symmetry is dynamically broken whenever $m_G \leq m_G^c$.) At these strong couplings the theory exhibits a nonperturbatively generated gap: the initially massless fermions and antifermions become massive via interaction with their own "gluon" field. Now the negative energy Dirac sea is only filled up to $E = -M$, with the positive energy states beginning at $E = +M$; i.e., the theory has a dynamically-generated, nonperturbative mass-gap $\Delta = 2M$. In addition, the quark condensate, which was zero when evaluated perturbatively because $m = 0$, is now nonzero.

Importantly, the nature of the solution of Eq. (4.3.12) also changes qualitatively when m_G is allowed to fall below it's critical value. It is in this way that DCSB continues to affect the hadronic spectrum even when the quarks have a small but nonzero current-mass.

4.3.2 *Munczek-Nemirovsky Model*

The gap equation in a model proposed more recently [MN83], which is able to represent a greater variety of the features of QCD while retaining much of the simplicity of the NJL model, is obtained from Eq. (4.3.1) by using

$$\frac{\mathcal{G}(k^2)}{k^2} = (2\pi)^4\, G\, \delta^4(k) \tag{4.3.17}$$

in Eq. (4.3.2) with the bare vertex, Eq. (4.3.3). Here G defines the model's mass-scale.

The gap equation is

$$i\gamma\cdot p\, A(p^2) + B(p^2) = i\gamma\cdot p + m + G\,\gamma_\mu \frac{-i\gamma\cdot p\, A(p^2) + B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}\,\gamma_\mu\,, \tag{4.3.18}$$

where again the renormalisation constants have been set equal to one but in this case because the model is ultraviolet finite; i.e., there are no infinities that must be regularised and subtracted. The gap equation yields the following two coupled equations:

$$A(p^2) = 1 + 2\,\frac{A(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \tag{4.3.19}$$

$$B(p^2) = m + 4\,\frac{B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}\,, \tag{4.3.20}$$

where we have set the mass-scale $G = 1$.

Consider the chiral limit equation for $B(p^2)$:

$$B(p^2) = 4\,\frac{B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}\,. \tag{4.3.21}$$

The existence of a $B \not\equiv 0$ solution; i.e., a solution that dynamically breaks chiral symmetry, requires

$$p^2 A^2(p^2) + B^2(p^2) = 4\,, \tag{4.3.22}$$

measured in units of G. Substituting this identity into equation Eq. (4.3.19) one finds

$$A(p^2) - 1 = \frac{1}{2}\,A(p^2) \;\Rightarrow\; A(p^2) \equiv 2\,, \tag{4.3.23}$$

which in turn entails

$$B(p^2) = 2\,\sqrt{1 - p^2}\,. \tag{4.3.24}$$

The physical requirement that the quark self energy be real in the spacelike region means that this solution is only acceptable for $p^2 \leq 1$. For $p^2 > 1$ one must choose the $B \equiv 0$ solution of Eq. (4.3.21), and on this domain we then find from Eq. (4.3.19) that

$$A(p^2) = 1 + \frac{2}{p^2\, A(p^2)} \;\Rightarrow\; A(p^2) = \tfrac{1}{2}\left(1 + \sqrt{1 + 8/p^2}\right) . \tag{4.3.25}$$

Putting this all together, the Munczek-Nemirovsky model exhibits a dynamical chiral symmetry breaking solution:

$$A(p^2) = \begin{cases} 2\,; & p^2 \leq 1 \\ \frac{1}{2}\left(1 + \sqrt{1 + 8/p^2}\right)\,; & p^2 > 1 \end{cases} \tag{4.3.26}$$

$$B(p^2) = \begin{cases} \sqrt{1 - p^2}\,; & p^2 \leq 1 \\ 0\,; & p^2 > 1\,. \end{cases} \tag{4.3.27}$$

which yields a nonzero quark condensate. Note that the dressed-quark self-energy is momentum dependent, as is the case in QCD.

It is important to observe that this solution is continuous and defined for all p^2, even $p^2 < 0$ which corresponds to timelike momenta, and furthermore

$$p^2\, A^2(p^2) + B^2(p^2) > 0\,,\; \forall\, p^2\,. \tag{4.3.28}$$

This last fact means that the quark described by this model is confined: the propagator does not exhibit a mass pole! We also note that there is no critical coupling in this model; i.e., the nontrivial solution for B always exists. This exemplifies a contemporary conjecture that theories with confinement always exhibit DCSB.

In the chirally asymmetric case the gap equation yields

$$A(p^2) = \frac{2\, B(p^2)}{m + B(p^2)}\,, \tag{4.3.29}$$

$$B(p^2) = m + \frac{4\,[m + B(p^2)]^2}{B(p^2)([m + B(p^2)]^2 + 4p^2)}\,. \tag{4.3.30}$$

The second is a quartic equation for $B(p^2)$ that can be solved algebraically with four solutions, available in a closed form, of which only one has the correct $p^2 \to \infty$ limit: $B(p^2) \to m$. Note that the equations and their solutions always have a smooth $m \to 0$ limit, a result owing to the persistence of the DCSB solution.

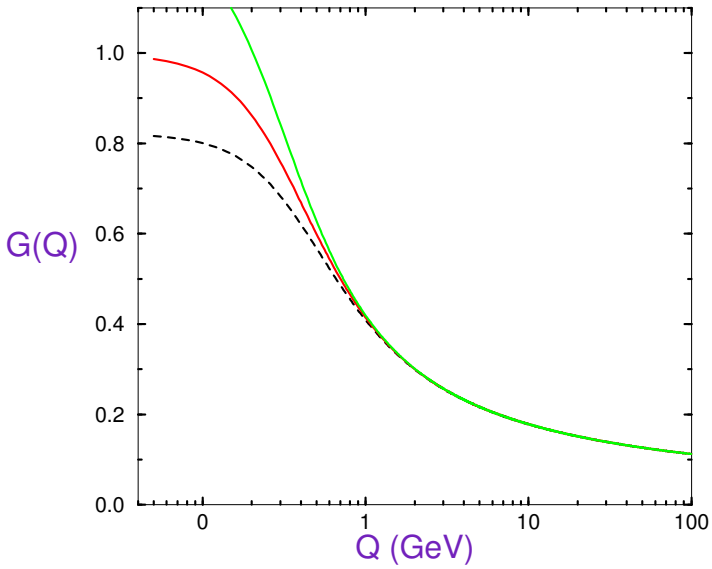

Fig. 4.3 Illustrative forms of $\mathcal{G}(Q)$: the behaviour of each agrees with perturbation theory for $Q > 1\,\mathrm{GeV}$. Three possibilities are canvassed in Sec. 4.3.3: $\mathcal{G}(Q=0) < 1$; $\mathcal{G}(Q=0) = 1$; and $\mathcal{G}(Q=0) > 1$.

4.3.3 *Renormalisation-Group-Improved Model*

Finally we have used the bare vertex, Eq. (4.3.3) and $\mathcal{G}(Q)$ depicted in Fig. 4.3, in solving the quark DSE in the chiral limit. If $\mathcal{G}(Q=0) < 1$ then $B(p^2) \equiv 0$ is the only solution. However, when $\mathcal{G}(Q=0) \geq 1$ the equation admits an energetically favoured $B(p^2) \not\equiv 0$ solution; i.e., if the coupling is large enough then even in the absence of a current-quark mass, contrary to Eq. (4.2.18), the quark acquires a mass *dynamically* and hence

$$\langle \bar{q} q \rangle \propto \int \frac{d^4p}{(2\pi)^4} \frac{B(p^2)}{p^2\, A(p^2)^2 + B(p^2)^2} \neq 0 \;\; \text{for } m = 0\,. \tag{4.3.31}$$

4.3.4 *Summary*

These examples identify a mechanism for DCSB in quantum field theory. The nonzero condensate provides a new, dynamically generated mass-scale and if its magnitude is large enough $[-\langle\bar{q}q\rangle^{1/3}$ need only be one order-of-magnitude larger than $m_u \sim m_d]$ it can explain the mass splitting between parity partners, and many other surprising phenomena in QCD. The models illustrate that DCSB is linked to the long-range behaviour of the fermion-fermion interaction. The same is true of confinement.

The question is then: How does the quark-quark interaction behave at large distances in QCD? It remains unanswered.

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