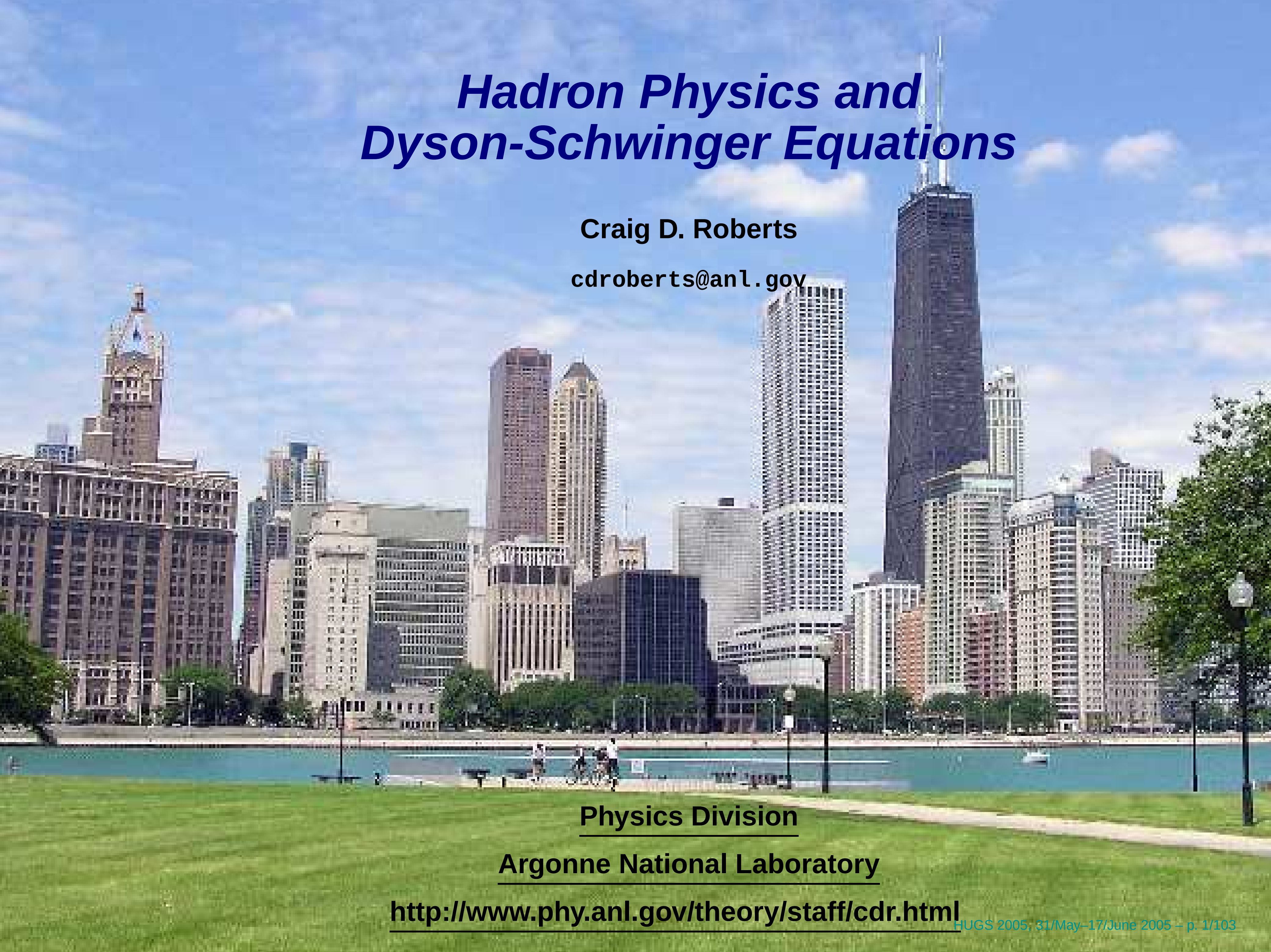

Hadron Physics and Dyson-Schwinger Equations

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Jan. Averages

- High: -1 C
- Low: -10 C

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Jan. Averages

- High: -1 C
- Low: -10 C

Jul. Averages

- High: 29 C
- Low: 20 C

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Physics Division

- ATLAS Tandem Linac Low Energy Nuclear Physics
- 33 PhD Scientific Staff
- Annual Budget: $20 million

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- 7 Staff
- 3 Postdocs
- 10 Special Term Appointees

Our research addresses the five key questions that comprise the Nation's scientific agenda. We place heavy emphasis on the prediction of phenomena accessible at Argonne's ATLAS facility, at JLab, and at other laboratories around the world; and on anticipating and planning for RIA. Additional research in the Group focuses on atomic physics, neutron physics, fundamental quantum mechanics and quantum computing. The pioneering development and use of massively parallel numerical simulations using hardware at Argonne and elsewhere is a major component of the Group's research.

Recommended Reading and Reviews

- C. D. Roberts and A. G. Williams, “Dyson-Schwinger equations and their application to hadronic physics,”
  Prog. Part. Nucl. Phys. **33** (1994) 477
  hep-ph/9403224
- C. D. Roberts and S. M. Schmidt, “Dyson-Schwinger equations: Density, temperature and continuum strong QCD,”
  Prog. Part. Nucl. Phys. **45** (2000) S1
  nucl-th/0005064
- R. Alkofer and L. von Smekal, “The infrared behavior of QCD Green’s functions: Confinement, dynamical symmetry breaking, and hadrons as relativistic bound states,”
  Phys. Rept. **353**, 281 (2001)
  hep-ph/0007355
- P. Maris and C. D. Roberts, “Dyson-Schwinger equations: A tool for hadron physics,”
  Int. J. Mod. Phys. E **12**, 297 (2003)
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Relativistic Quantum Field Theory

- A theoretical understanding of the phenomena of Hadron Physics requires the use of the full machinery of relativistic quantum field theory. Relativistic quantum field theory is the ONLY way to reconcile quantum mechanics with special relativity.
- Relativistic quantum field theory is based on the relativistic quantum mechanics of Dirac.
- Relativistic quantum mechanics predicts the existence of antiparticles; i.e., the equations of relativistic quantum mechanics admit *negative energy* solutions. However, once one allows for particles with negative energy, then particle number conservation is lost:
  $E_{\rm system} = E_{\rm system} + (E_{p_1} + E_{\bar{p}_1}) + \ldots$ *ad infinitum*
- However, this is a fundamental problem for relativistic quantum mechanics – Few particle systems can be studied in relativistic quantum mechanics but the study of (infinitely) many bodies is difficult. No general theory currently exists.

Not all Poincaré transformations commute with the Hamiltonian. Hence, a Poincaré transformation from one frame to another can change the number of particles. Therefore a solution of the N body problem in one frame is generally insufficient to study the typical scattering process encountered at JLab.

Relativistic Quantum Field Theory

- Relativistic quantum field theory is an answer. The fundamental entities are fields, which can simultaneously represent an uncountable infinity of particles.

  e.g., neutral scalar: $$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)\mathrm{e}^{-ik\cdot x} + a^\dagger(k)\mathrm{e}^{ik\cdot x}\right] \quad (1)$$

  Hence, the nonconservation of particle number is not a problem. This is crucial because key observable phenomena in hadron physics are essentially connected with the existence of *virtual* particles.

- Relativistic quantum field theory has its own problems, however. For the mathematician, the question of whether a given relativistic quantum field theory is rigorously well defined is *unsolved*.

- All relativistic quantum field theories admit analysis in perturbation theory. Perturbative renormalisation is a well-defined procedure and has long been used in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD).

- A rigorous definition of a theory, however, means proving that the theory makes sense *nonperturbatively*. This is equivalent to proving that all the theory's renormalisation constants are nonperturbatively well-behaved.

Relativistic Quantum Field Theory

- Hadron Physics involves QCD. While it makes excellent sense perturbatively: Gross, Politzer, Wilczek – Nobel Prize 2004 – *Asymptotic Freedom*. It is not known to be a rigorously well-defined theory. Hence it cannot truly be said to be THE theory of the strong interaction (hadron physics).
- Nevertheless, physics does not wait on mathematics. Physicists make assumptions and explore their consequences. Practitioners assume that QCD is (somehow) well-defined and follow where that leads us.
- Experiment: explore and map the hadron physics landscape with well-understood probes, such as the electron at JLab.
- Theory: employ established mathematical tools, and refine and invent others in order to use the Lagangian of QCD to predict what should be observable real-world phenomena.
- A key current aim of the worlds' hadron physics programmes in experiment and theory is to determine whether there are any obvious contradictions with what we can actually *prove* in QCD. Hitherto, there are none.
- Interplay between Experiment and Theory is the engine of discovery and progress. The *Discovery Potential* of both is high. Much learnt in the last five years and I expect that many surprises remain in Hadron Physics.

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Relativistic Quantum Field Theory

- In modern approaches to relativistic quantum field theory the primary elements are Green Functions or Schwinger Functions. These are the objects that naturally arise in the Functional Integral formulation of a theory.
- Introduce this concept via the Green function of the Dirac operator from relativistic quantum mechanics. To do this we're going to need some background on notation and conventions in relativistic quantum mechanics.

NB. Spin-$\frac{1}{2}$ particles do *not* have a classical analogue. Pauli, in his paper on the exclusion principle: Quantum Spin is a "classically nondescribable two-valuedness." Problem: Configuration space of quantum spin is quite different to that accessible with a classical spinning top.

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Notation

- *contravariant* four-vector:

$$x^\mu := (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)\,. \tag{2}$$

$c = 1 = \hbar$, and the conversion between length and energy is just:

$$1\,\mathrm{fm} = 1/(0.197327\,\mathrm{GeV}) = 5.06773\,\mathrm{GeV}^{-1}\,. \tag{3}$$

- *covariant* four-vector is obtained by changing the sign of the spatial components of the contravariant vector:

$$x_\mu := (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z) = g_{\mu\nu} x^\nu\,, \tag{4}$$

where the metric tensor is

$$g_{\mu\nu} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]\,. \tag{5}$$

Notation

- Contracted product of two four-vectors is

$$(a,b) := g_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu \, ; \tag{6}$$

i.e, a contracted product of a covariant and contravariant four-vector. The Poincaré-invariant length of any vector is $\boxed{x^2 := (x,x) = t^2 - \vec{x}^2}$.

- Momentum vectors are similarly defined:

$$p^\mu = (E, p_x, p_y, p_z) = (E, \vec{p}) \tag{7}$$

and

$$(p,k) = p_\mu k^\mu = E_p E_k - \vec{p} \cdot \vec{k} \, . \tag{8}$$

Likewise, a mixed coordinate-momentum contraction:

$$(x,p) = tE - \vec{x} \cdot \vec{p} \, . \tag{9}$$

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Notation

- Momentum *operator*

$$p^\mu := i\frac{\partial}{\partial x_\mu} = (i\frac{\partial}{\partial t}, \frac{1}{i}\vec{\nabla}) =: i\nabla^\mu \tag{10}$$

transforms as a contravariant four-vector. Poincaré invariant analogue of the Laplacian is

$$\partial^2 := -p^\mu p_\mu = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\,. \tag{11}$$

- Contravariant four-vector associated with the electromagnetic field:

$$A^\mu(x) = (\Phi(x), \vec{A}(x)). \tag{12}$$

Electric and magnetic field strengths obtained from $\boxed{F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\mu;}$ for example,

$$\vec{E}_i = F^{0i}\,;\ \text{i.e. } \vec{E} = -\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\,. \tag{13}$$

Similarly, $B^i = \epsilon^{ijk}F^{jk}$, $j,k = 1,2,3$.
Analogous definitions hold in QCD for the chromomagnetic field strengths.

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Dirac Matrices

- The Dirac matrices are *indispensable* in a manifestly Poincaré covariant description of particles with spin (intrinsic angular momentum).
- The Dirac matrices are defined by the Clifford Algebra (an identity matrix is implicit on the r.h.s.)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\, g^{\mu\nu}\,, \tag{14}$$

- One common 4×4 representation is [each entry represents a 2×2 matrix]

$$\gamma^0 = \left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{array}\right],\; \vec{\gamma} = \left[\begin{array}{cc} \mathbf{0} & \vec{\boldsymbol{\sigma}} \\ -\vec{\boldsymbol{\sigma}} & \mathbf{0} \end{array}\right], \tag{15}$$

where $\vec{\sigma}$ are the usual Pauli matrices:

$$\sigma^1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right],\; \sigma^2 = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right],\; \sigma^3 = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right], \tag{16}$$

and $\mathbf{1} := \mathrm{diag}[1,1]$. Clearly: $\gamma_0^\dagger = \gamma_0$; and $\vec{\gamma}^\dagger = -\vec{\gamma}$. NB. These properties are *not* specific to this representation; e.g., $\gamma^1\gamma^1 = -\mathbf{1}_{4\times 4}$, *for any* representation of the Clifford algebra.

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Dirac Matrices

- In discussing spin, two combinations of Dirac matrices frequently appear:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \, , \; \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 \, . \tag{17}$$

NB. $\boxed{\gamma^5 \, \sigma^{\mu\nu} = \frac{i}{2} \, \epsilon^{\mu\nu\rho\sigma} \, \sigma_{\rho\sigma} ,}$ with $\epsilon^{\mu\nu\rho\sigma}$ the completely antisymmetric Lèvi-Civita tensor: $\epsilon^{0123} = +1$, $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$.

- The Dirac matrix γ_5 plays a special role in the discussion of parity and chiral symmetry, two key aspects of the Standard Model. In the representation we're using,

$$\gamma^5 = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} . \tag{18}$$

Furthermore
$$\{\gamma_5, \gamma^\mu\} = 0 \; \Rightarrow \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5 \quad \& \quad \gamma_5^\dagger = \gamma_5 \, . \tag{19}$$

- Parity, Chiral Symmetry, and the relation between them, each play a special role in Hadron Physics.

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Dirac Matrices

- The “slash” notation is a frequently used shorthand:

$$\gamma^\mu A_\mu =: \not{A} = \gamma^0 A^0 - \vec{\gamma}\cdot\vec{A}\,, \tag{20}$$
$$\gamma^\mu p_\mu =: \not{p} = \gamma^0 E - \vec{\gamma}\cdot\vec{p}\,, \tag{21}$$
$$\gamma^\mu \boldsymbol{p}_\mu =: i\not{\nabla} \equiv i\not{\partial} = i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} = i\gamma^\mu\frac{\partial}{\partial x^\mu}\,. \tag{22}$$

- The following identities are important in evaluating the cross-sections.

$$\mathrm{tr}\,\gamma_5 = 0\,, \qquad \mathrm{tr}\,\mathbf{1} = 4\,, \tag{23}$$
$$\mathrm{tr}\,\not{a}\not{b} = 4(a,b)\,, \qquad \mathrm{tr}\,\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4\,[(a_1,a_2)(a_3,a_4) - (a_1,a_3)(a_2,a_4) + (a_1,a_4)(a_2,a_3)]\,, \tag{24}$$
$$\mathrm{tr}\,\not{a}_1\ldots\not{a}_n = 0\,,\ \text{for n odd}\,, \qquad \mathrm{tr}\,\gamma_5\not{a}\,\not{b} = 0\,, \tag{25}$$
$$\mathrm{tr}\,\gamma_5\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4i\epsilon_{\alpha\beta\gamma\delta}\,a^\alpha b^\beta c^\gamma d^\delta\,, \qquad \gamma_\mu\not{a}\gamma^\mu = -2\not{a}\,, \tag{26}$$
$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a,b)\,, \qquad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}\,, \tag{27}$$

All follow from the fact that the Dirac matrices satisfy the Clifford algebra.
Exercises: Prove these relations using Eq. (14) and exploiting $\mathrm{tr}\,AB = \mathrm{tr}\,BA$.

Relativistic Quantum Mechanics

- Unification of special relativity (Poincaré covariance) and quantum mechanics took some time. Questions still remain as to a practical implementation of an Hamiltonian formulation of the relativistic quantum mechanics of interacting systems.
- Poincaré group has ten generators: the six associated with the Lorentz transformations (rotations and boosts) and the four associated with translations.
- Quantum mechanics describes the time evolution of a system with interactions. That evolution is generated by the Hamiltonian.
- However, if the theory is formulated with an interacting Hamiltonian then boosts will almost always fail to commute with the Hamiltonian. Hence, the state vector calculated in one momentum frame will not be kinematically related to the state in another frame. That makes a new calculation necessary in every frame.
- Hence the discussion of scattering, which takes a state of momentum p to another state with momentum p', is problematic. (See, e.g., B.D. Keister and W.N. Polyzou (1991), "Relativistic Hamiltonian dynamics in nuclear and particle physics," *Adv. Nucl. Phys*. **20**, 225.)

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Dirac Equation

- Dirac equation is starting point for Lagrangian formulation of quantum field theory for fermions. For a noninteracting fermion

$$[i\partial\!\!\!/ - m]\,\psi = 0\,, \tag{28}$$

where $\psi(x)$ is the fermion's "spinor" – a four component column vector, with each component spacetime dependent.

- In an external electromagnetic field the fermion's wave function obeys

$$[i\partial\!\!\!/ - eA\!\!\!/ - m]\,\psi = 0\,, \tag{29}$$

obtained, as usual, via "minimal substitution:" $\boldsymbol{p}^\mu \to \boldsymbol{p}^\mu - eA^\mu$ in Eq. (28). The Dirac operator is a matrix-valued differential operator.

- These equations have a manifestly Poincaré covariant appearance. A proof of covariance is given in the early chapters of: Bjorken, J.D. and Drell, S.D. (1964), *Relativistic Quantum Mechanics* (McGraw-Hill, New York).

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Free Particle Solutions

- Insert plane waves in free particle Dirac equation:
  $$\psi^{(+)}(x) = \mathrm{e}^{-i(k,x)}\, u(k)\,,\; \psi^{(-)}(x) = \mathrm{e}^{+i(k,x)}\, v(k)\,,$$
  and thereby obtain ...
  $$(\not k - m)\, u(k) = 0\,,\; (\not k + m)\, v(k) = 0\,. \tag{30}$$
  Here there are two qualitatively different types of solution, corresponding to positive and negative energy: k & $-k$. (Appreciation of physical reality of negative energy solutions led to prediction of antiparticles.)
- Assume particle's mass is nonzero; work in rest frame:
  $$(\gamma^0 - \mathbf{1})\, u(m,\vec{0}) = 0\,,\; (\gamma^0 + \mathbf{1})\, v(m,\vec{0}) = 0\,. \tag{31}$$
  There are clearly (remember the form of γ^0) two linearly-independent solutions of each equation:
  $$u^{(1),(2)}(m,\vec{0}) = \left(\begin{array}{c} 1\\ 0\\ 0\\ 0 \end{array}\right), \left(\begin{array}{c} 0\\ 1\\ 0\\ 0 \end{array}\right),\; v^{(1),(2)}(m,\vec{0}) = \left(\begin{array}{c} 0\\ 0\\ 1\\ 0 \end{array}\right), \left(\begin{array}{c} 0\\ 0\\ 0\\ 1 \end{array}\right). \tag{32}$$

Positive Energy Free Particle

- Solution in arbitrary frame can be obtained via a Lorentz boost. However, simpler to observe that

$$(\not{k} - m)\,(\not{k} + m) = k^2 - m^2 = 0\,, \tag{33}$$

(The last equality is valid for real, on-shell particles.)
It follows that for arbitrary k^μ and positive energy ($E > 0$), the canonically normalised spinor is

$$u^{(\alpha)}(k) \;=\; \frac{\not{k} + m}{\sqrt{2m(m+E)}}\, u^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \left(\dfrac{E+m}{2m}\right)^{1/2} & \phi^\alpha(m,\vec{0}) \\ \dfrac{\sigma \cdot k}{\sqrt{2m(m+E)}} & \phi^\alpha(m,\vec{0}) \end{pmatrix}, \tag{34}$$

with the two-component spinors, obviously to be identified with the fermion's spin in the rest frame (the only frame in which spin has its naive meaning)

$$\phi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \phi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{35}$$

Negative Energy Free Particle

- For negative energy: $\hat{E} = -E > 0$,

$$v^{(\alpha)}(k) = \frac{-\not{k} + m}{\sqrt{2m(m+\hat{E})}} v^{(\alpha)}(m,\vec{0}) = \left(\begin{array}{cc} \dfrac{\sigma \cdot k}{\sqrt{2m(m+\hat{E})}} & \chi^{\alpha}(m,\vec{0}) \\ \left(\dfrac{\hat{E}+m}{2m}\right)^{1/2} & \chi^{\alpha}(m,\vec{0}) \end{array} \right), \tag{36}$$

with $\chi^{(\alpha)}$ obvious analogues of $\phi^{(\alpha)}$ in Eq. (35).

- NB. For $\vec{\boldsymbol{k}} \sim \boldsymbol{0}$ (rest frame) the lower component of the positive energy spinor is small, as is the upper component of the negative energy spinor $\Rightarrow$ Poincaré covariance, which requires the four component form, becomes important with increasing $|\vec{\boldsymbol{k}}|$; indispensable for $|\vec{\boldsymbol{k}}| \sim \boldsymbol{m}$.

NB. Solving $\vec{\boldsymbol{k}} \neq \boldsymbol{0}$ equations this way works because it is clear that there are two, and only two, linearly-independent solutions of the momentum space free-fermion Dirac equations, Eqs. (30), and, for the homogeneous equations, any two covariant solutions with the correct limit in the rest-frame must give the correct boosted form.

Conjugate Spinor

- In quantum field theory, as in quantum mechanics, one needs a conjugate state to define an inner product.

  For fermions in Minkowski space that conjugate is $\boxed{\bar{\psi}(x) := \psi^\dagger(x)\gamma^0}$, and

$$\bar{\psi}(i \overleftarrow{\not\partial} + m) = 0\,. \tag{37}$$

- This yields the following free particle spinors in momentum space (using $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, relation that is particularly important in the discussion of intrinsic parity)

$$\bar{u}^{(\alpha)}(k) = \bar{u}^{(\alpha)}(m,\vec{0})\,\frac{\not k + m}{\sqrt{2m(m+E)}} \tag{38}$$

$$\bar{v}^{(\alpha)}(k) = \bar{v}^{(\alpha)}(m,\vec{0})\,\frac{-\not k + m}{\sqrt{2m(m+E)}}\,, \tag{39}$$

- Orthonormalisation

$$\begin{array}{ll} \bar{u}^{(\alpha)}(k)\,u^{(\beta)}(k) = \delta_{\alpha\beta} & \bar{u}^{(\alpha)})(k)\,v^{(\beta)}(k) = 0 \\ \bar{v}^{(\alpha)})(k)\,v^{(\beta)}(k) = -\delta_{\alpha\beta} & \bar{v}^{(\alpha)})(k)\,u^{(\beta)}(k) = 0 \end{array}\,. \tag{40}$$

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Positive Energy Projection Operator

- Can now construct positive energy projection operators. Consider

$$\Lambda_+(k) \quad := \quad \sum_{\alpha=1,2} u^{(\alpha)}(k) \otimes \bar{u}^{(\alpha)}(k)\,. \tag{41}$$

Plain from the orthonormality relations, Eqs. (40), that

$$\Lambda_+(k)\, u^{(\alpha)}(k) = u^{(\alpha)}(k)\,,\;\; \Lambda_+(k)\, v^{(\alpha)}(k) = 0\,. \tag{42}$$

- Now, since $\sum_{\alpha=1,2} u^{(\alpha)}(m,\vec{0}) \otimes \bar{u}^{(\alpha)}(m,\vec{0}) = \left(\begin{array}{cc} \mathbf{1} & 0 \\ 0 & 0 \end{array}\right) = \frac{\mathbf{1}+\gamma^0}{2}$, then

$$\Lambda_+(k) \quad = \quad \frac{1}{2m(m+E)}\,(\not k + m)\,\frac{\mathbf{1}+\gamma^0}{2}\,(\not k + m)\,. \tag{43}$$

- Noting that for $k^2 = m^2$; i.e., on shell,
$(\not k + m)\,\gamma^0\,(\not k + m) = 2E\,(\not k + m)$, $(\not k + m)\,(\not k + m) = 2m\,(\not k + m)$,
one finally arrives at the simple closed form:

$$\Lambda_+(k) = \frac{\not k + m}{2\,m}\,. \tag{44}$$

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Negative Energy Projection Operator

- The negative energy projection operator is

$$\Lambda_-(k) \quad := \quad -\sum_{\alpha=1,2} v^{(\alpha)}(k) \otimes \bar{v}^{(\alpha)}(k) = \frac{-\not{k} + m}{2\,m}\,. \tag{45}$$

- The projection operators have the following characteristic and important properties:

$$\Lambda_\pm^2(k) \quad = \quad \Lambda_\pm(k)\,, \tag{46}$$

$$\mathrm{tr}\,\Lambda_\pm(k) \quad = \quad 2\,, \tag{47}$$

$$\Lambda_+(k) + \Lambda_-(k) \quad = \quad \mathbf{1}\,. \tag{48}$$

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Green-Functions/Propagagors

- The Dirac equation is a partial differential equation.
A general method for solving such equations is to use a Green function, which is the inverse of the differential operator that appears in the equation.
The analogy with matrix equations is obvious and can be exploited heuristically.
- Dirac equation, Eq. (29): $\boxed{[i\not\partial_x - e\not A(x) - m]\,\psi(x) = 0\,,}$ yields the wave function for a fermion in an external electromagnetic field.
- Consider the operator obtained as a solution of the following equation

$$[i\not\partial_{x'} - e\not A(x') - m]\,S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \tag{49}$$

- Obviously if, at a given spacetime point x, $\psi(x)$ is a solution of Eq. (29), then

$$\psi(x') := \int d^4x\,S(x',x)\,\psi(x) \tag{50}$$

$$\text{is a solution of}\ldots\; [i\not\partial_{x'} - e\not A(x') - m]\,\psi(x') = 0\,; \tag{51}$$

i.e., $S(x',x)$ has *propagated* the solution at x to the point x'.

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i.e., $S(x',x)$ has *propagated* the solution at x to the point x'.

Analogue of Huygens Principle in Wave Mechanics

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Green-Functions/Propagagors

- This approach is practical because all physically reasonable external fields can only be nonzero on a compact subdomain of spacetime.
- Therefore the solution of the complete equation is transformed into solving for the Green function, which can then be used to propagate the free-particle solution, already found, to arbitrary spacetime points.
- However, obtaining the *exact* form of $S(x', x)$ is *impossible* for all but the simplest cases

  (see, e.g., Dittrich, W. and Reuter, M. (1985), *Effective Lagrangians in Quantum Electrodynamics* (Springer Verlag, Berlin); Dittrich, W. and Reuter, M. (1985), *Selected Topics in Gauge Theories* (Springer Verlag, Berlin).)
- This is where and why perturbation theory so often rears its not altogether handsome head.

Free Fermion Propagator

- In the absence of an external field the Green Function equation, Eq. (49), becomes

$$[i\not\partial_{x'} - m]\, S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \tag{55}$$

- Assume a solution of the form:

$$S_0(x',x) = S_0(x'-x) = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-i(p,x'-x)}\, S_0(p)\,, \tag{56}$$

so that substituting yields

$$(\not p - m)\, S_0(p) = \mathbf{1}\,;\ \text{i.e.,}\ S_0(p) = \frac{\not p + m}{p^2 - m^2}\,. \tag{57}$$

- To obtain the result in configuration space one must adopt a prescription for handling the on-shell singularities in $S(p)$ at $p^2 = m^2$.
  - That convention is tied to the boundary conditions applied to Eq. (55).
  - An obvious and physically sensible definition of the Green function is that it should propagate positive-energy-fermions and -antifermions forward in time but not backwards in time, and vice versa for negative energy states.

Feynman's Fermion Propagator

- The wave function for a positive energy free-fermion is

$$\psi^{(+)}(x) = u(p)\,\mathrm{e}^{-i(p,x)}\,. \tag{58}$$

The wave function for a positive-energy antifermion is the charge-conjugate of the negative-energy fermion solution ($C = i\gamma^2\gamma^0$ and $(\cdot)^{\mathrm{T}}$ denotes matrix transpose):

$$\psi_c^{(+)}(x) = C\,\gamma^0\,\left(v(p)\,\mathrm{e}^{i(p,x)}\right)^* = C\,\bar{v}(p)^{\mathrm{T}}\,\mathrm{e}^{-i(p,x)}\,, \tag{59}$$

- Follows from properties of spinors and projection operators that our physically sensible $S_0(x'-x)$ must contain only positive-frequency components for $x_0' - x_0 > 0$; i.e., in this case it must be proportional to $\Lambda_+(p)$.
  - Exercise: Verify this.
- Can ensure this via a small modification of the denominator of Eq. (57), with $\boldsymbol{\eta} \to 0^+$ at the end of all calculations:

$$S_0(p) = \frac{\not p + m}{p^2 - m^2} \to \frac{\not p + m}{p^2 - m^2 + i\boldsymbol{\eta}}\,. \tag{60}$$

(This prescription defines the **Feynman** propagator.)

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Feynman's Fermion Propagator

- To demonstrate (Energy $\omega(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$):

$$S_0(x'-x) = \int \frac{d^3p}{(2\pi)^3}\, \mathrm{e}^{i\vec{p}\cdot(\vec{x}\,'-\vec{x})}\, \frac{1}{2\,\omega(\vec{p})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}$$
$$\times \left[\mathrm{e}^{-ip^0(x_0'-x_0)}\, \frac{\not p + m}{p^0 - \omega(\vec{p}) + i\eta} - \mathrm{e}^{-ip^0(x_0'-x_0)}\, \frac{\not p + m}{p^0 + \omega(\vec{p}) - i\eta}\right], \quad (61)$$

- Use Cauchy's Theorem; focus on first term of the sum inside the square brackets:
  - Integrand has a pole in the fourth quadrant of the complex p^0-plane.
  - $x_0' - x_0 > 0 \ldots$ evaluate p^0 integral by considering a contour closed by a semicircle of radius $R \to \infty$ in the lower half of the complex p^0-plane
  - The closed contour is oriented clockwise so that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(x_0'-x_0)}\, \frac{\not p + m}{p^0 - \omega(\vec{p}) + i\eta^+} &= (-)\,i\;\mathrm{e}^{-ip^0(x_0'-x_0)}(\not p + m)\Big|_{p^0=\omega(\vec{p})-i\eta} \\ &= -i\,\mathrm{e}^{-i\omega(\vec{p})(x_0'-x_0)}\,(\gamma^0\omega(\vec{p}) - \gamma\cdot\vec{p} + m) \\ &= -i\,\mathrm{e}^{-i\omega(\vec{p})(x_0'-x_0)}\, 2m \boxed{\Lambda_+(p)}\,. \quad (6\end{aligned}$$

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Feynman's Fermion Propagator

- $x_0' - x_0 < 0$:
Contour must be closed in the upper half plane but therein the integrand is analytic and hence the result is zero.

$$\text{Thus} \ldots \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \, \mathrm{e}^{-ip^0(x_0'-x_0)} \boxed{i \, \frac{\not p + m}{p^0 - \omega(\vec{p}) + i\eta^+}}$$

$$= \theta(x_0' - x_0) \, \mathrm{e}^{-i\omega(\vec{p})(x_0'-x_0)} \, 2m \, \Lambda_+(\vec{p}) \,. \tag{63}$$

NB. Projection operator is truly only a function of $\vec{p}$ because $p^0 = \omega(\vec{p})$.

- Second term in the brackets (similar reasoning):

$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \, \mathrm{e}^{-ip^0(x_0'-x_0)} \boxed{-i \, \frac{\not p + m}{p^0 + \omega(\vec{p}) - i\eta^+}} = \theta(x_0 - x_0') \, \mathrm{e}^{+i\omega(\vec{p})(x_0'-x_0)} \, 2m \, \Lambda_-(-\vec{p}) \,. \tag{64}$$

- Complete result [changing variables $\vec{p} \to -\vec{p}$ in Eq. (64) & $(\tilde{p}^\mu) := (\omega(\vec{p}), \vec{p})$]

$$iS_0(x'-x) = \int \frac{d^3p}{(2\pi)^3} \, \frac{m}{\omega(\vec{p})} \left[\theta(x_0' - x_0) \, \mathrm{e}^{-i(\tilde{p}, x'-x)} \, \Lambda_+(\vec{p}) + \theta(x_0 - x_0') \, \mathrm{e}^{i(\tilde{p}, x'-x)} \Lambda_-(\vec{p}) \right] \tag{65}$$

Feynman's Propagator: Alternative Representation

- Another useful representation is obtained merely by observing that

$$\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x_0)}\,\frac{\not p+m}{p^0-\omega(\vec p)+i\eta^+} = \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x)}\,\frac{\gamma^0\omega(\vec p)-\vec\gamma\cdot\vec p+m}{p^0-\omega(\vec p)+i\eta^+}$$

$$= \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x)}\,2m\Lambda_+(\vec p)\,\frac{1}{p^0-\omega(\vec p)+i\eta^+}\,, \qquad (6$$

$$\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x_0)}\,\frac{\not p+m}{p^0+\omega(\vec p)-i\eta^+} = \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x)}\,\frac{-\gamma^0\omega(\vec p)-\vec\gamma\cdot\vec p+m}{p^0+\omega(\vec p)-i\eta^+}$$

$$= \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(x_0'-x)}\,2m\Lambda_-(-\vec p)\,\frac{1}{p^0+\omega(\vec p)-i\eta^+}\,, \qquad (6$$

- Hence, $S_0(x'-x)=\int\frac{d^4p}{(2\pi)^4}\,\mathrm{e}^{-i(p,x'-x)}\,S_0(p)$,

$$S_0(p)=\frac{m}{\omega(\vec p)}\left[\Lambda_+(\vec p)\,\frac{1}{p^0-\omega(\vec p)+i\eta}-\Lambda_-(-\vec p)\,\frac{1}{p^0+\omega(\vec p)-i\eta}\right]\,, \qquad (68)$$

This representation provides single Fourier amplitude for $S_0(x_0'-x_0)$; i.e., an alternative to Eq. (60): indispensable in making connection between covariant and time-ordered perturbation theory – the second term generates the Z-diagrams in loop integrals.

Green Function: Interacting Theory

- Eq. (49), Green function for a fermion in an external electromagnetic field:

$$[i\not\partial_{x'} - e\not A(x') - m]\, S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,, \tag{69}$$

A closed form solution of this equation is impossible in all but the simplest field configurations. Is there, nevertheless, a way to construct an approximate solution that can systematically be improved?

- One Answer: Perturbation Theory – rewrite the equation:

$$[i\not\partial_{x'} - m]\, S(x',x) = \mathbf{1}\,\delta^4(x'-x) + e\not A(x')\, S(x',x)\,, \tag{70}$$

which, as is easily seen by substitution (Verify This), is solved by

$$\begin{aligned}
S(x',x) &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S(y,x)\\
&= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S_0(y-x)\\
&\quad + e^2\int d^4y_1\int d^4y_2\, S_0(x'-y_1)\not A(y_1)\, S_0(y_1-y_2)\not A(y_2)\, S_0(y_2-x)\\
&\quad + \ldots
\end{aligned} \tag{71}$$

Green Function: Interacting Theory

- This perturbative expansion of the full propagator in terms of the free propagator provides an archetype for perturbation theory in quantum field theory.
  - One obvious application is the scattering of an electron/positron by a Coulomb field, which is an example explored in Sec. 2.5.3 of Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York).
  - Equation (71) is a first example of a Dyson-Schwinger equation.
- This Green function has the following interpretation
  1. It creates a positive energy fermion (antifermion) at spacetime point x;
  2. Propagates the fermion to spacetime point x'; i.e., forward in time;
  3. Annihilates this fermion at x'.

  The process can equally well be viewed as

  1. The creation of a negative energy antifermion (fermion) at spacetime point x';
  2. Propagation of the antifermion to the spacetime point x; i.e., backward in time;
  3. Annihilation of this antifermion at x.

  Other propagators have similar interpretations.

Exercises

- Prove these relations for on-shell fermions:

$$\begin{aligned}(\not k + m)\,\gamma^0\,(\not k + m) &= 2E\,(\not k + m)\,,\\ (\not k + m)\,(\not k + m) &= 2m\,(\not k + m)\,.\end{aligned}$$

- Obtain the Feynman propagator for the free-field Klein Gordon equation:

$$(\partial_x^2 + m^2)\phi(x) = 0\,,$$

in forms analogous to Eqs. (65), (68).

Anything Troubling You?

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Functional Integrals

- Local gauge theories are the keystone of contemporary hadron and high-energy physics. QCD is a local gauge theory. Such theories are difficult to quantise because the gauge dependence is an extra non-dynamical degree of freedom that must be dealt with. The modern approach is to quantise the theories using the method of functional integrals. Good references:
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York);
  - Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin).

  Functional Integration replaces canonical second-quantisation. NB. In general mathematicians do not regard local gauge theory functional integrals as well-defined.
- As a background, we'll review a path integral formulation of quantum mechanics.

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Path Integral in Quantum Mechanics

- Begin with a state $(q_1, q_2, \ldots, q_N)$ at time t. Probability of obtaining state $(q'_1, q'_2, \ldots, q'_N)$ at time t' is (remember, the time evolution operator in quantum mechanics is $\exp[-iHt]$, where H is the system's Hamiltonian):

$$\langle q'_1, q'_2, \ldots, q'_N; t' | q_1, q_2, \ldots, q_N; t\rangle = \lim_{n\to\infty} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} dq_\alpha(t_i) \int \prod_{i=1}^{n+1} \frac{dp_\alpha(t_i)}{2\pi}$$

$$\times \exp\left[i\epsilon \sum_{j=1}^{n+1} \left\{ p_\alpha(t_j) \frac{1}{\epsilon}[q_\alpha(t_j) - q_\alpha(t_{j-1})] - H(p(t_j), \frac{q_{t_j} + q_{t_j - 1}}{2}) \right\} \right] \quad (72)$$

  where $t_j = t + j\epsilon$, $\epsilon = (t' - t)/(n+1)$, $t_0 = t$, $t_{n+1} = t'$.

- A compact notation is commonly introduced to represent this expression:

$$\langle q' t' | q t\rangle^J = \int [dq] \int [dp] \; \mathrm{e}^{i \int_t^{t'} d\tau \, [p(\tau)\dot{q}(\tau) - H(\tau) + J(\tau) q(\tau)]} \quad (73)$$

  where $J(t)$ is a classical external "source." NB. The $J = 0$ exponent is nothing but the Lagrangian: $\boxed{L = p\dot{q} - H}$. This representation of the Green function for a quantum mechanical system owes to Feynman. Details may be found in Feynman, R.P. and Hibbs, A.R., *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965). It is also described in modern textbooks.

Path Integral in Quantum Mechanics

- Heisenberg's formulation of quantum mechanics: operators evolve in time, not the state vectors, whose values are fixed at a given initial time. It follows from the previous formulae that the time-ordered product of n Heisenberg position operators can be expressed

$$\begin{aligned}&\langle q't'|T\{Q(t_1)\ldots Q(t_n)\}|qt\rangle\\&\quad=\int [dq]\int [dp]\, q(t_1)\, q(t_2)\ldots q(t_n)\ \mathrm{e}^{i\int_t^{t'} d\tau\,[p(\tau)\dot{q}(\tau)-H(\tau)]}\end{aligned} \tag{74}$$

  - NB. The time-ordered product ensures that the operators appear in chronological order, right to left.
  - NB. $Q(t_i)$ are operators. $q(t_i)$ are c-numbers.
- Any expectation value measurable in quantum mechanics can be written in this way.

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Path Integral in Quantum Mechanics

- Consider a source that "switches on" at t_i and "switches off" at t_f, $t < t_i < t_f < t'$, then
$$\langle q' t' | q t \rangle^J = \int dq_i dq_f \, \langle q' t' | q_f t_f \rangle \, \langle q_f t_f | q_i t_i \rangle^J \, \langle q_i t_i | q t \rangle \, . \tag{75}$$

- Alternative: introduce a complete set of energy eigenstates to resolve the Hamiltonian and write
$$\langle q' t' | q t \rangle^J = \sum_n \langle q' | \phi_n \rangle \, \mathrm{e}^{-iE_n(t'-t)} \, \langle \phi_n | q \rangle \overset{\substack{t' \to -i\infty \\ t \to i\infty}}{=} \langle q' | \phi_0 \rangle \, \mathrm{e}^{-iE_0(t'-t)} \, \langle \phi_0 | q \rangle \, ; \tag{76}$$
i.e., in these limits the transition amplitude is dominated by the ground state.

- It follows from Eqs. (75), (76) that the object
$$W[J] := \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{\langle q' t' | q t \rangle^J}{\mathrm{e}^{-iE_0(t'-t)} \, \langle q' | \phi_0 \rangle \, \langle \phi_0 | q \rangle} = \int dq_i dq_f \, \langle \phi_0 | q_f t_f \rangle \, \langle q_f t_f | q_i t_i \rangle^J \, \langle q_i t_i | \phi_0 \rangle \tag{77}$$
i.e., it is the ground-state to ground-state transition amplitude (survival probability) in the presence of the external source J. Plainly $W[J=0] = 1$.

Functional Derivative

- Functional Derivative: $\dfrac{\delta}{\delta J(t)}$, is defined analogously to the derivative of a function. It means: write $J(t) \to J(t) + \epsilon(t)$; expand the functional in $\epsilon(t)$; and identify the leading order coefficient in the expansion as the functional derivative. Thus for

$$H_n[J] = \int dt'\, J(t')^n \,, \tag{78}$$

$$\begin{aligned} \delta H_n[J] &= \delta \int dt'\, J(t')^n = \int dt'\, [J(t') + \epsilon(t')]^n - \int dt'\, J(t')^n \\ &= \int dt'\, n\, J(t')^{n-1}\, \epsilon(t') + [\ldots] \end{aligned} \tag{79}$$

$$\Rightarrow \frac{\delta H_n[J]}{\delta J(t)} = n\, J(t)^{n-1} \,. \tag{80}$$

One employs the definition
$$\frac{\delta J(t)}{\delta J(t')} = \delta(t - t') \,. \tag{81}$$

Eq. (80) is the "product rule." Plainly, there is a very close analogy between functional and ordinary differentiation. With a little care, the functional differentiation of complicated functionals is straightforward.

Path Integral in Quantum Mechanics

- $W[J]$ is called a *Generating Functional*.
- Apparent now that, with $t_f > t_1 > t_2 > \ldots > t_m > t_i$,

$$\left.\frac{\delta^m W[J]}{\delta J(t_1)\,\ldots\,\delta J(t_1)}\right|_{J=0} = i^m \int dq_i dq_f \,\langle\phi_0|q_f t_f\rangle\,\langle q_f t_f|T\{Q(t_1)\ldots Q(t_n)\}|q_i t_i\rangle\,\langle q_i t_i|\phi_0\rangle, \quad (82)$$

is the ground state (vacuum) expectation value of a time ordered product of Heisenberg position operators.

- The analogues of these expectation values in quantum field theory are the Green functions.

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Scalar Quantum Field

- Consider a scalar field $\phi(t,x)$. This is customary because it reduces the number of indices that must be carried through the calculation.
- Suppose that a large but compact domain of space is divided into N cubes of volume ϵ^3 and label each cube by an integer α.
- Define the coordinate and momentum via

$$q_\alpha(t) := \phi_\alpha(t) = \frac{1}{\epsilon^3}\int_{V_\alpha} d^3x\,\phi(t,x)\,,\ \dot q_\alpha(t) := \dot\phi_\alpha(t) = \frac{1}{\epsilon^3}\int_{V_\alpha} d^3x\,\frac{\partial\phi(t,x)}{\partial t}\,; \tag{83}$$

  i.e., as the spatial averages over the cube denoted by α.
- Classical dynamics of the field ϕ is described by a Lagrangian:

$$\mathrm{L}(t) = \int d^3x\,L(t,x) \to \sum_{\alpha=1}^{N}\epsilon^3\,L_\alpha(\dot\phi_\alpha(t),\phi_\alpha(t),\phi_{\alpha\pm s}(t))\,, \tag{84}$$

  where the dependence on $\phi_{\alpha\pm s}(t)$; i.e., the coordinates in the neighbouring cells, is necessary in order to express spatial derivatives in the Lagrangian density, $L(x)$.
- Define canonical conjugate momentum as in classical field theory

$$p_\alpha(t) := \frac{\partial L}{\partial\dot\phi_\alpha(t)} = \epsilon^3\,\frac{\partial L_\alpha}{\partial\dot\phi_\alpha(t)} =: \epsilon^3\pi_\alpha(t)\,, \tag{85}$$

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Scalar Quantum Field

- Hamiltonian: $\mathrm{H} = \sum_\alpha p_\alpha(t)\,\dot{q}_\alpha(t) - L(t) \quad =: \quad \sum_\alpha \epsilon^3\, H_\alpha\,,$ (86)

$$H_\alpha(\pi_\alpha(t),\phi_\alpha(t),\phi_{\alpha\pm s}(t)) \quad = \quad \pi_\alpha(t)\,\dot{\phi}_\alpha(t) - L_\alpha\,. \tag{87}$$

- Field theoretical equivalent of quantum mechanical transition amplitude, Eq. (72):

$$\int [\mathcal{D}\phi] \int [\mathcal{D}\pi]\, \exp\left\{ i \int_t^{t'} d\tau \int d^3x\, \left[\pi(\tau,\vec{x}) \frac{\partial\phi(\tau,\vec{x})}{\partial\tau} - H(\tau,\vec{x}) \right] \right\}$$

$$:= \lim_{n\to\infty,\epsilon\to 0^+} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} d\phi_\alpha(t_i) \int \prod_{i=1}^{n} \epsilon^3 \frac{d\pi_\alpha(t_i)}{2\pi}$$

$$\times \exp\left[i \sum_{j=1}^{n+1} \epsilon \sum_\alpha \epsilon^3 \left\{ \pi_\alpha(t_j) \frac{\phi_\alpha(t_j) - \phi_\alpha(t_{j-1})}{\epsilon} \right.\right.$$

$$\left.\left. - H_\alpha\left(\pi_\alpha(t_j), \frac{\phi_\alpha(t_j)+\phi_\alpha(t_{j-1})}{2}, \frac{\phi_{\alpha\pm s}(t_j)+\phi_{\alpha\pm s}(t_{j-1})}{2} \right) \right\} \right] \tag{88}$$

As classically, $\pi(t,\vec{x}) = \partial L(t,\vec{x})/\partial\dot{\phi}(t,\vec{x})$ and its average over a spacetime cube is just $\pi_\alpha(t)$. Equation (88): amplitude describing transition from initial field configuration $\phi_\alpha(t_0) := \phi_\alpha(t)$ to a final configuration $\phi_\alpha(t_{n+1}) := \phi_\alpha(t')$.

Generating Functional: Scalar Quantum Field

- In quantum field theory all physical quantities can be obtained from Green functions, which are vacuum-to-vacuum transition amplitudes.
- The physical or interacting vacuum is the analogue of the true ground state in quantum mechanics.
- As in quantum mechanics, the fundamental quantity is the generating functional:

$$W[J] := \frac{1}{\mathcal{N}} \int [\mathcal{D}\phi][\mathcal{D}\pi] \, \mathrm{e}^{i \int d^4x \, [\pi(x)\dot{\phi}(x) - H(x) + \frac{1}{2} i\eta\phi^2(x) + J(x)\phi(x)]}, \tag{89}$$

where $\mathcal{N}$ is chosen so that $W[0] = 1$, and a real-time limit is implemented and made meaningful by adding the $\eta \to 0^+$ term.

- It is immediately apparent that (Schwinger, 1951)

$$G(x_1, x_2, \ldots, x_n) := \frac{1}{i^n} \frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\bigg|_{J=0} = \frac{\langle\tilde{0}|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle}, \tag{90}$$

where $|\tilde{0}\rangle$ is the physical vacuum. $G(x_1, x_2, \ldots, x_n)$ is the *complete* n-point Green function for the scalar quantum field theory: "complete" means $G(x_1, x_2, \ldots, x_n)$ includes contributions from products of lower-order Green functions; i.e., disconnected diagrams.

Connected Green Functions

- Useful to have systematic procedure for *a priori* elimination of disconnected parts from n-point Green function because recalculation of $m < n$-point Green functions, is inefficient. A *connected* n-point Green function is given by

$$G_c(x_1, x_2, \ldots, x_n) = (-i)^{n-1} \left. \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\right|_{J=0}, \tag{91}$$

where generating functional for connected Green functions, $Z[J]$, defined via

$$W[J] =: \exp\{iZ[J]\}\,. \tag{92}$$

- Illustration for simple case:

$$\begin{aligned}
G_c(x_1,x_2) &= (-i) \left.\frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = - \left.\frac{\delta^2 \ln W[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} \\
&= - \frac{\delta}{\delta J(x_1)} \left[\frac{1}{W[J]}\,\frac{\delta W[J]}{\delta J(x_2)}\right]\Bigg|_{J=0} \\
&= + \frac{1}{W^2[J]}\,\frac{\delta W[J]}{\delta J(x_1)}\,\frac{\delta W[J]}{\delta J(x_2)}\Bigg|_{J=0} - \frac{1}{W[J]}\,\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\Bigg|_{J=0} \\
&= i\frac{\langle\tilde{0}|\hat{\phi}(x_1)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle}\, i\frac{\langle\tilde{0}|\hat{\phi}(x_2)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} - i^2\,\frac{\langle\tilde{0}|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \\
&= -G(x_1)\,G(x_2) + G(x_1,x_2)\,.
\end{aligned} \tag{93}$$

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Lagrangian Formulation

- Double functional integral employed above $\int[\mathcal{D}\phi]\int[\mathcal{D}\pi]$ is cumbersome, especially since it involves the field variable's canonical conjugate momentum. Consider therefore a Hamiltonian density of the form
$$H(x) = \frac{1}{2}\pi^2(x) + f[\phi(x), \vec{\nabla}\phi(x)]\,. \tag{94}$$
In this case Eq. (89) involves
$$\begin{aligned}\int[\mathcal{D}\pi]\,\mathrm{e}^{i\int d^4x\,[-\frac{1}{2}\pi^2(x)+\pi(x)\dot\phi(x)]} &= \mathrm{e}^{\displaystyle i\int d^4x\,[\dot\phi(x)]^2}\int[\mathcal{D}\pi]\,\mathrm{e}^{\displaystyle -\frac{i}{2}\int d^4x\,[\pi(x)-\dot\phi(x)]^2} \\ &= \mathrm{e}^{\displaystyle \{i\int d^4x\,[\dot\phi(x)]^2\}} \times \mathrm{N}\,, \leftarrow \text{ simply a constant}\end{aligned} \tag{95}$$
NB. Only Gaussian integral can be evaluated exactly. Hence, with Eq. (94),
$$W[J] = \frac{\mathrm{N}}{\mathcal{N}}\int[\mathcal{D}\phi]\,\mathrm{e}^{\displaystyle i\int d^4x\,[L(x)+\frac{1}{2}i\eta\phi^2(x)+J(x)\phi(x)]} \tag{96}$$
Classical Lagrangian density for a scalar field is $L(x) = L_0(x) + L_I(x)$
$$L_0(x) = \frac{1}{2}[\partial_\mu\phi(x)\,\partial^\mu\phi(x) - m^2\phi^2(x)]\,, \tag{97}$$
with $L_I(x)$ some functional of $\phi(x)$ (usually independent of derivatives of the field). Matches Eq. (94) $\Rightarrow$ Eq. (96) can be used to define the quantum field theory.

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QFT: Free Scalar Field

- Free scalar field: $L_I \equiv 0$, $L_0(x)$ given in Eq. (97), so that the generating functional is *formally*

$$W_0[J] = \frac{1}{\bar{\mathcal{N}}} \int [\mathcal{D}\phi] \, \mathrm{e}^{\, i \int d^4x \, \left[L_0(x) + i\eta\phi^2(x)\right] + J(x)\phi(x)]} \tag{98}$$

- Explicitly, this means

$$W_0[J] = \lim_{\epsilon \to 0^+} \frac{1}{\bar{\mathcal{N}}} \int \prod_\alpha d\phi_\alpha \, \exp\left\{ i \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \, \frac{1}{2} \, \phi_\alpha \, K_{\alpha\beta} \, \phi_\beta + \sum_\alpha \epsilon^4 \, J_\alpha \phi_\alpha \right\}, \tag{99}$$

α, β label spacetime hypercubes of volume ϵ^4, $K_{\alpha\beta}$ is any matrix satisfying

$$\lim_{\epsilon \to 0^+} K_{\alpha\beta} = [-\partial^2 - m^2 + i\eta] \, \delta^4(x-y)\,, \tag{100}$$

where $\alpha \stackrel{\epsilon \to 0^+}{\to} x$, $\beta \stackrel{\epsilon \to 0^+}{\to} y$ and $\sum_\alpha \epsilon^4 \stackrel{\epsilon \to 0^+}{\to} \int d^4x$; i.e., $K_{\alpha\beta}$ is any matrix whose continuum limit is the inverse of the Feynman propagator for a free scalar field.

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QFT: Free Scalar Field

- Recall now that for matrices whose real part is positive definite

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} dx_i \, \exp\{-\frac{1}{2}\sum_{i,j=1}^{n} x_i\, A_{ij}\, x_j + \sum_{i=1}^{n} b_i\, x_i\} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\{\frac{1}{2}\sum_{i,j=1}^{n} b_i\, (A^{-1})_{ij}\, b_j\}$$
$$= \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left\{\frac{1}{2}\boldsymbol{b}^{\rm t} A^{-1} \boldsymbol{b}\right\} . \quad (101)$$

- Hence Eq. (99) yields

$$W_0[J] = \lim_{\epsilon\to 0+} \frac{1}{\bar{\mathcal{N}}'} \frac{1}{\sqrt{\det A}} \exp\{\frac{1}{2}\sum_{\alpha} \epsilon^4 \sum_{\beta} \epsilon^4\, J_\alpha \frac{1}{i\epsilon^8} (K^{-1})_{\alpha\beta}\, J_\beta\} , \quad (102)$$

where, obviously, $\sum_{\gamma} K_{\alpha\gamma}\, (K^{-1})_{\gamma\beta} = \delta_{\alpha\beta}$. (103)

- Almost as obviously, consistency of limits requires

$$\lim_{\epsilon\to 0+} \frac{1}{\epsilon^4}\, \delta_{\alpha\beta} = \delta^4(x-y) , \quad \lim_{\epsilon\to 0+} \sum_{\alpha} \epsilon^4 = \int d^4x \quad (104)$$

QFT: Free Scalar Field

- Define $\mathrm{O}(x,y) := \lim_{\epsilon\to 0^+} \dfrac{1}{\epsilon^8}(K^{-1})_{\alpha\beta}$
- Then continuum limit of Eq. (103) can be understood:

$$\lim_{\epsilon\to 0^+} \sum_\gamma \epsilon^4\, K_{\alpha\gamma}\, \frac{1}{\epsilon^8}(K^{-1})_{\gamma\beta} = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^4}\,\delta_{\alpha\beta}$$

$$\Rightarrow \quad \int d^4w\,[-\partial_x^2 - m^2 + i\eta]\,\delta^4(x-w)\,O(w,y) = \delta^4(x-y)$$

$$\therefore \quad [-\partial_x^2 - m^2 + i\eta]\,O(x,y) = \delta^4(x-y)\,. \tag{105}$$

- Hence $O(x,y) = \Delta_0(x-y)$; i.e., the Feynman propagator for a free scalar field:

$$\Delta_0(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{e}^{-i(q,x-y)}\,\frac{1}{q^2-m^2+i\eta}\,. \tag{106}$$

(NB. This makes plain the fundamental role of the "$i\eta^+$" prescription in Eq. (60): it ensures convergence of the expression defining the functional integral.)

- Combining all, the continuum limit of Eq. (102) is

$$W[J] = \frac{1}{\hat{\mathcal{N}}}\,\mathrm{e}^{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\,\Delta_0(x-y)\,J(y)}\,. \tag{107}$$

QFT: Self-interacting Scalar Field

- $L_I[\phi(x)] \neq 0$ provides for a self-interacting scalar field theory (we subsequently omit the constant, nondynamical normalisation factor):

$$W[J] = \int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \left[L_0(x) + L_I(x) + J(x)\phi(x) \right] \right\}$$

$$= \exp\left[i \int d^4x \, L_I\left(\frac{\delta}{i\delta J(x)} \right) \right] \int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \left[L_0(x) + J(x)\phi(x) \right] \right\}$$

$$= \exp\left[i \int d^4x \, L_I\left(\frac{\delta}{i\delta J(x)} \right) \right] \exp\left\{ -\frac{i}{2} \int d^4x \int d^4y \, J(x) \, \Delta_0(x-y) \, J(y) \right\}$$

where

$$\exp\left[i \int d^4x \, L_I\left(\frac{\delta}{i\delta J(x)} \right) \right] := \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[L_I\left(\frac{\delta}{i\delta J(x)} \right) \right]^n . \qquad (108)$$

- Equation (108) is the basis for a perturbative evaluation of all possible Green functions for the theory.
- Example: complete 2-point Green function in the theory defined by

$$\boxed{L_I(x) = -\frac{\lambda}{4!} \phi^4(x)}.$$

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QFT: Self-interacting Scalar Field

- At leading-order, the generating functional yields

$$W[0] = \left\{1 - i\frac{\lambda}{4!}\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4\right\} \exp\left\{-\frac{i}{2}\int d^4u \int d^4v\, J(u)\,\Delta_0(u-v)\,J(v)\right\}\Bigg|_{J=0}$$

$$\Rightarrow W[0] = 1 - i\frac{\lambda}{4!}\int d^4x\, 3\,[i\Delta_0(0)]^2\,. \tag{109}$$

- The 2-point function is

$$\begin{aligned}\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)} \quad &= \quad -i\Delta_0(x_1-x_2) + i\,\frac{\lambda}{8}\int d^4x\,[i\Delta_0(0)]^2[i\Delta_0(x_1-x_2)] \\ &\quad +i\,\frac{\lambda}{2}\int d^4x\,[i\Delta_0(0)][i\Delta_0(x_1-x)][i\Delta_0(x-x_2)]\,.\end{aligned} \tag{110}$$

- Now using, Eq. (90), and restoring the normalisation we find

$$\begin{aligned}G(x_1,x_2) &= \frac{1}{i^2}\,\frac{1}{W[0]}\,\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)} \\ &= \quad i\Delta_0(x_1-x_2) - i\,\frac{\lambda}{2}\int d^4x\,[i\Delta_0(0)][i\Delta_0(x_1-x)][i\Delta_0(x-x_2)]\end{aligned} \tag{111}$$

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Note on the Perturbative Vacuum

- In obtaining Eq. (111) we used

$$\frac{\langle\tilde{0}|\hat{\phi}(x)|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} := G(x)|_{J=0} = \frac{1}{i}\,\frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0} = 0 \tag{112}$$

  (You can easily verify this.)

- Equation (112) means that the vacuum is *trivial* in perturbation theory.
- That is the reason why the *complete* 2-point Green function in Eq. (111) does not contain any disconnected parts.
- NB. Equation (112) is the simplest demonstration of the fact that dynamical symmetry breaking is a phenomenon inaccessible in perturbation theory. The vacuum has no structure in perturbation theory.

Fermions in Quantum Field Theory

- Fermionic fields do not have a classical analogue: classical physics does not contain anticommuting fields. In order to treat fermions using functional integrals one must employ Grassmann variables.
- The standard source for a rigorous discussion of Grassmann algebras is Berezin, F.A. (1966), *The Method of Second Quantization* (Academic Press, New York).
  Here we will only review some necessary ideas.
- The Grassmann algebra G_N is generated by the set of N elements, $\theta_1 , \ldots , \theta_N$, which satisfy the anticommutation relations

$$\{\theta_i, \theta_j\} = 0\,,\; i,j = 1,2,\ldots,N\,. \tag{113}$$

  Clearly, $\theta_i^2 = 0$ for $i = 1,\ldots,N$.
- Moreover, the elements $\{\theta_i\}$ provide the source for the basis vectors of a 2^n-dimensional space, spanned by the monomials:

$$1,\; \theta_1,\ldots,\theta_N,\; \theta_1\theta_2,\ldots\theta_{N-1}\theta_N,\ldots,\; \theta_1\theta_2\ldots\theta_N\,; \tag{114}$$

  i.e., G_N is a 2^N-dimensional vector space.

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Fermions in Quantum Field Theory

- Obviously, any element $f(\theta) \in G_N$ can be written

$$
\begin{aligned}
f(\theta) \quad = \quad & f_0 + \sum_{i_1} f_1(i_1)\,\theta_{i_1} + \sum_{i_1,i_2} f_2(i_1,i_2)\,\theta_{i_1}\theta_{i_2} + \ldots \\
& + \sum_{i_1,i_2,\ldots,i_N} f_N(i_1,i_2,\ldots,i_N)\,\theta_1\theta_2\ldots\theta_N\,, \qquad (115)
\end{aligned}
$$

  $f_p(i_1,i_2,\ldots,i_p)$ are unique if chosen to be fully antisymmetric for $p \geq 2$.

- Both “left” and “right” derivatives are defined on G_N. They are linear operators. Hence it suffices to specify their operation on the basis elements:

$$
\frac{\partial}{\partial\theta_s}\,\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p} = \delta_{si_1}\theta_{i_2}\ldots\theta_{i_p} - \delta_{si_2}\theta_{i_1}\ldots\theta_{i_p} + \ldots + (-)^{p-1}\delta_{si_p}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_{p-1}} \qquad (116)
$$

$$
\begin{aligned}
& \theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}\,\frac{\overleftarrow{\partial}}{\partial\theta_s} \\
& = \theta_{i_1}\ldots\theta_{i_{p-1}} - \delta_{si_{p-1}}\theta_{i_1}\ldots\theta_{i_{p-2}}\theta_{i_p} + \ldots + (-)^{p-1}\delta_{si_1}\theta_{i_2}\theta_{i_p} \qquad (117)
\end{aligned}
$$

  The operation on a general element, $f(\theta) \in G_N$, is easily obtained.
  Obviously, $\dfrac{\partial}{\partial\theta_1}\dfrac{\partial}{\partial\theta_2}\,f(\theta) = -\dfrac{\partial}{\partial\theta_2}\dfrac{\partial}{\partial\theta_1}f(\theta)$.

Fermions in Quantum Field Theory

- A definition of integration requires the introduction of Grassmannian line elements: $d\theta_i$, $i = 1, \ldots, N$. These elements also satisfy Grassmann algebras:

$$\{d\theta_i, d\theta_j\} = 0 = \{\theta_i, d\theta_j\}\,,\;\; i,j = 1,2,\ldots N\,. \tag{118}$$

- The integral calculus is completely defined by the following two identities:

$$\int d\theta_i = 0\,,\;\; \int d\theta_i\,\theta_i = 1\,,\;\; i = 1,2,\ldots,N\,. \tag{119}$$

- For example, it is straightforward to prove, using Eq. (115),

$$\int d\theta_N \ldots d\theta_1\, f(\theta) = N!\, f_N(1,2,\ldots,N)\,. \tag{120}$$

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Fermions in Quantum Field Theory

- In standard integral calculus a change of integration variables is often used to simplify an integral. That operation can also be defined in the present context. Consider a nonsingular matrix (K_{ij}), $i, j = 1, \ldots, N$, and define new Grassmann variables $\xi_1, \ldots, \xi_N$ via

$$\theta_i = \sum_{i=j}^{N} K_{ij}\,\xi_j\,. \tag{121}$$

  - With the definition $d\theta_i = \sum_{j=1}^{N} (K^{-1})_{ji}\, d\xi_j$

    one guarantees $\int d\theta_i\,\theta_j = \delta_{ij} = \int d\xi_i\,\xi_j$.
  - It follows immediately that

$$\theta_1\theta_2\ldots\theta_N = (\det K)\,\xi_1\xi_2\ldots\xi_N \tag{122}$$

$$d\theta_N\, d\theta_{N-1}\ldots d\theta_1 = (\det K^{-1})\, d\xi_N\, d\xi_{N-1}\ldots d\xi_1\,, \tag{123}$$

    NB. This is inverted w.r.t. c-numbers.
  - Hence

$$\int d\theta_N \ldots d\theta_1\, f(\theta) = (\det K^{-1}) \int d\xi_N \ldots d\xi_1\, f(\theta(\xi))\,. \tag{124}$$

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Fermion Gaussian Integral

- In analogy with scalar field theory, for fermions one expects to encounter integrals of the type

$$I := \int d\theta_N \dots d\theta_1 \, \exp\left\{ \sum_{i,j=1}^{N} \theta_i A_{ij} \theta_j \right\} \,, \tag{125}$$

  where (A_{ij}) is an antisymmetric matrix. NB. Any symmetric part of the matrix, A, cannot contribute. (Exercise: Verify.)

- Assume for the moment that A is a real matrix. Then there is an orthogonal matrix S ($SS^{\rm t} = I$) for which

$$S^{\rm t} A S = \left[\begin{array}{ccccc} 0 & \lambda_1 & 0 & 0 & \dots \\ -\lambda_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_2 & \dots \\ 0 & 0 & -\lambda_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array} \right] =: \tilde{A} \,. \tag{126}$$

  Consequently, applying transformation $\theta_i = \sum_{i=1}^{N} S_{ij}\, \xi_j$ and using Eq. (124):

$$I = \int d\xi_N \dots d\xi_1 \, \exp\left\{ \sum_{i,j=1}^{N} \xi_i \tilde{A}_{ij} \xi_j \right\} \,. \tag{127}$$

Fermion Gaussian Integral

- It is now plain that

$$I = \begin{cases} \int d\xi_N \ldots d\xi_1 \;\exp\left\{2\left[\lambda_1 \xi_1 \xi_2 + \lambda_2 \xi_3 \xi_4 + \ldots + \lambda_{N/2} \xi_{N-1} \xi_N\right]\right\} \\ \qquad\qquad = 2^{N/2} \lambda_1 \lambda_2 \ldots \lambda_N \,, \;\; N \,\text{even} \\ \int d\xi_N \ldots d\xi_1 \;\exp\left\{2\left[\lambda_1 \xi_1 \xi_2 + \lambda_2 \xi_3 \xi_4 + \ldots + \lambda_{(N-1)/2} \xi_{N-2} \xi_{N-1}\right]\right\} \\ \qquad\qquad = 0 \,, \;\; N \,\text{odd} \end{cases} \tag{128}$$

  i.e., since $\det A = \det \tilde{A}$,

$$\boxed{I = \sqrt{\det 2A}\,.} \tag{129}$$

- Equation (129) is valid for any real matrix, A.
  Hence, by the analytic function theorem, it is also valid for any complex matrix A.

Grassmann Algebra with Involution

- Lagrangian density associated with the Dirac equation involves a field $\bar{\psi}$, which plays the role of a conjugate to ψ.
  - If ψ is vector in G_N, then need a conjugate space in which $\bar{\psi}$ is defined.
  - Hence it is necessary to define $\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N$ such that the operation $\theta_i \leftrightarrow \bar{\theta}_i$ is an involution of the algebra onto itself with the following properties:

$$i) \quad \overline{(\bar{\theta}_i)} = \theta_i \quad ii) \quad \overline{(\theta_i \theta_j)} = \bar{\theta}_j\, \bar{\theta}_i \quad iii) \quad \overline{\lambda\, \theta_i} = \lambda^*\, \bar{\theta}_i\,,\ \lambda \in \mathbb{C}\,. \tag{130}$$

- Elements of Grassmann algebra with involution: θ_1, θ_2, ..., θ_N, $\bar{\theta}_1$, $\bar{\theta}_2$ ..., $\bar{\theta}_N$, each anticommuting with every other.
- Defining integration via obvious analogy with Eq. (119) $\Rightarrow$ for any matrix B:

$$\int d\bar{\theta}_N d\theta_N \ \ldots \ d\bar{\theta}_1 d\theta_1 \ \exp\left\{ - \sum_{i,j=1}^{N} \bar{\theta}_i B_{ij} \theta_j \right\} = \det B\,, \tag{131}$$

- cf. Analogous result for commuting real numbers, Eq. (101):

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} dx_i \ \exp\{-\pi \sum_{i,j=1}^{N} x_i\, A_{ij}\, x_j\} = \frac{1}{\sqrt{\det A}}\,. \tag{132}$$

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Fermions in Quantum Field Theory

- To describe a fermionic quantum field the preceding analysis must be generalised to the case of infinitely many generators. We'll just review a plausible description.
- Suppose functions $\{u_n(x)\,,\; n = 0,\ldots,\infty\}$ are a complete, orthonormal set that span a given Hilbert space. Consider the Grassmann function

$$\theta(x) := \sum_{n=0}^{\infty} u_n(x)\, \theta_n\,, \tag{133}$$

where $\{\theta_n\}$ are Grassmann variables. Clearly $\{\theta(x), \theta(y)\} = 0$.

- Elements $\theta(x)$ are the generators of the "Grassman algebra" G. In complete analogy with Eq. (115), any element of G can be written uniquely as

$$f = \sum_{n=0}^{\infty} \int dx_1 dx_2 \ldots dx_N\, \theta(x_1)\theta(x_2)\ldots\theta(x_N)\, f_n(x_1, x_2, \ldots, x_N)\,. \tag{134}$$

$f_n(x_1, x_2, \ldots, x_N)$ are fully antisymmetric functions of their arguments.

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Fermions in Quantum Field Theory

- Another analogy, the left-functional-derivative is defined (cf. Eq. (116).):

$$\frac{\delta}{\delta\theta(x)}\,\theta(x_1)\theta(x_2)\ldots\theta(x_n) =$$
$$\delta(x-x_1)\,\theta(x_2)\ldots\theta(x_n) - \ldots + (-)^{n-1}\delta(x-x_n)\,\theta(x_1)\ldots\theta(x_{n-1})$$

  Straightforward analogy for right-functional-derivative.

- Moreover, one can extend the definition of integration.
  Denoting $[\mathcal{D}\theta(x)] := \lim_{N\to\infty} d\theta_N \ldots d\theta_2 d\theta_1$, consider the Gaussian integral

$$I := \int [\mathcal{D}\theta(x)]\; \exp\left\{\int dx dy\, \theta(x) A(x,y) \theta(y)\right\} \quad (135)$$

  where, clearly, only the antisymmetric part of $A(x,y)$ can contribute. Define

$$A_{ij} := \int dx dy\, u_i(x) A(x,y) u_j(y)\,, \quad (136)$$

  then $I = \lim_{N\to\infty} \int d\theta_N \ldots d\theta_2 d\theta_1\; \exp\left\{\sum_{i=1}^{N} \theta_i A_{ij} \theta_j\right\}$

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Fermions in Quantum Field Theory

- Now, using Eq. (129),

$$\boxed{I = \lim_{N\to\infty} \sqrt{\det 2A_N},}$$

where A_N is the $N \times N$ matrix in Eq. (136).

- This provides a definition for the formal result:

$$I = \int [\mathcal{D}\theta(x)] \, \exp\left\{ \int dx dy \, \theta(x) A(x,y) \theta(y) \right\} = \sqrt{\mathrm{Det}\, 2A}\,, \tag{137}$$

NB. We will subsequently identify functional equivalents of matrix operations as proper nouns; e.g., "det $\to$ Det."

- The result is independent of the basis vectors since all such vectors are unitarily equivalent and the determinant is cylic.
  - This means that a new basis is always related to another basis via $\boldsymbol{u}' = U\boldsymbol{u}$, with $UU^\dagger = \boldsymbol{I}$. Transforming to a new basis therefore introduces a modified exponent, now involving the matrix $UAU^\dagger$, but the result is unchanged because $\det UAU^\dagger = \det A$.

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Fermion Fields and Involution

- In quantum field theory one employs a Grassmann algebra with an involution. In this case, defining the functional integral via

$$[\mathcal{D}\bar{\theta}(x)][\mathcal{D}\theta(x)] := \lim_{N\to\infty} d\bar{\theta}_N d\theta_N \ldots d\bar{\theta}_2 d\theta_2\, d\bar{\theta}_1 d\theta_1\,, \tag{138}$$

one arrives immediately at a generalisation of Eq. (131)

$$\int [\mathcal{D}\bar{\theta}(x)][\mathcal{D}\theta(x)] \exp\left\{-\int dxdy\, \bar{\theta}(x)\, B(x,y)\, \theta(x)\right\} = \mathrm{Det} B\,. \tag{139}$$

- The relation

$$\ln\det B = \mathrm{tr}\ln B\,, \tag{140}$$

is valid for any nonsingular, finite dimensional matrix. It has a generalisation often used in analysing quantum field theories with fermions. It enables a representation of the fermionic determinant as part of the quantum field theory's action via

$$\mathrm{Det} B = \exp\left\{\mathrm{Tr}\,\mathrm{Ln} B\right\}\,. \tag{141}$$

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Observations on Functionals

- We note that for an integral operator $O(x,y)$

$$\mathrm{Tr}\, O(x,y) := \int d^4x\, \mathrm{tr}\, O(x,x)\,, \tag{142}$$

which is an obvious analogy to the definition for finite-dimensional matrices.

- Moreover, a functional of an operator, whenever it is well-defined, is obtained via the function's power series; i.e., if

$$f(x) = f_0 + f_1\, x + f_2\, x^2 + [\ldots]\,, \tag{143}$$

then

$$f[O(x,y)] = f_0\, \delta^4(x-y) + f_1\, O(x,y) + f_2 \int d^4w\, O(x,w) O(w,y) + [\ldots]\,. \tag{144}$$

Generating Functional: Free Dirac Fields

- The Lagragian density for the free Dirac field is

$$L_0^{\psi}(x) = \int d^4x\, \bar\psi(x)\,(i\partial\!\!\!/ - m)\,\psi(x)\,. \tag{145}$$

- Consider therefore the functional integral

$$W[\bar\xi,\xi] = \int [\mathcal{D}\bar\psi(x)][\mathcal{D}\psi(x)]\, \mathrm{e}^{\,i\int d^4x\,\left[\bar\psi(x)\,\left(i\partial\!\!\!/ - m + i\eta^+\right)\psi(x) + \bar\psi(x)\xi(x) + \bar\xi(x)\psi(x)\right]}\,. \tag{146}$$

$\bar\psi(x)$, $\psi(x)$ are identified with generators of G. NB. Minor additional complication: spinor degree-of-freedom is implicit; i.e., to be explicit, one should write

$$\prod_{r=1}^{4}[\mathcal{D}\bar\psi_r(x)]\prod_{s=1}^{4}[\mathcal{D}\psi_s(x)]. \tag{147}$$

Only adds a finite matrix degree-of-freedom to the problem, so that "$\mathrm{Det}\,A$" will mean both a functional *and* a matrix determinant. In Eq. (146) we have also introduced anticommuting sources: $\bar\xi(x)$, $\xi(x)$, which are also elements in the Grassmann algebra, G.

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Generating Functional: Free Dirac Fields

- To evaluate free-field generating functional Gaussian integral, write

$$O(x,y) = (i\partial\!\!\!/ \; - m + i\eta^+)\delta^4(x-y) \tag{148}$$

and observe that the solution of $\int d^4w\, O(x,w)\, P(w,y) = \boldsymbol{I}\, \delta^4(x-y)$ i.e., the inverse of operator $O(x,y)$ is (see Eq. 55) precisely free-fermion propagator:

$$P(x,y) = S_0(x-y)\,. \tag{149}$$

- Hence we can rewrite Eq. (146) in the form

$$W[\bar\xi,\xi] = \int [\mathcal{D}\bar\psi(x)][\mathcal{D}\psi(x)]\, \mathrm{e}^{\, i \int d^4x d^4y\; [\bar\psi'(x) O(x,y)\psi'(y) - \bar\xi(x) S_0(x-y)\xi(y)]}\,, \tag{150}$$

where
$$\begin{aligned} \bar\psi'(x) &:= \bar\psi(x) + \int d^4w\, \bar\xi(w)\, S_0(w-x)\,,\\ \psi(x) &:= \psi(x) + \int d^4w\, S_0(x-w)\,\xi(w)\,. \end{aligned} \tag{151}$$

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Generating Functional: Free Dirac Fields

- Clearly, $\bar{\psi}'(x)$ and $\psi'(x)$ are still in G and hence related to original variables by unitary transformation. Thus changing to "primed" variables introduces unit Jacobian and so

$$\begin{aligned}
W[\bar{\xi},\xi] &= \mathrm{e}^{-i\int d^4x d^4y\, \bar{\xi}(x)\, S_0(x-y)\, \xi(y)} \\
&\quad \times \int [\mathcal{D}\bar{\psi}'(x)][\mathcal{D}\psi'(x)]\, \mathrm{e}^{i\int d^4x d^4y\, \bar{\psi}'(x) O(x,y) \psi'(y)} \\
&= \mathrm{Det}[-iS_0^{-1}(x-y)]\, \mathrm{e}^{-i\int d^4x d^4y\, \bar{\xi}(x)\, S_0(x-y)\, \xi(y)} \\
W[\bar{\xi},\xi] &= \frac{1}{\mathcal{N}_0^{\psi}}\, \mathrm{e}^{-i\int d^4x d^4y\, \bar{\xi}(x)\, S_0(x-y)\, \xi(y)} ,
\end{aligned} \tag{152}$$

  where $\mathcal{N}_0^{\psi} := \mathrm{Det}[iS_0(x-y)]$.

- Clearly: $\mathcal{N}_0^{\psi}\ W[\bar{\xi},\xi]\big|_{\bar{\xi}=0=\xi} = 1$.

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Complete 2-point Free Green Function

- The 2 point Green function for the free-fermion quantum field theory is now easily obtained:

$$\left.\frac{\delta^2 W[\bar{\xi},\xi]}{i\delta\bar{\xi}(x)\,(-i)\delta\xi(y)}\right|_{\bar{\xi}=0=\xi} = \frac{\langle 0|T\{\hat{\psi}(x)\hat{\bar{\psi}}(y)\}|0\rangle}{\langle 0|0\rangle}$$
$$= \int [\mathcal{D}\bar{\psi}(x)][\mathcal{D}\psi(x)]\,\psi(x)\bar{\psi}(y)\,\mathrm{e}^{\,i\int d^4x\,\bar{\psi}(x)\,\left(i\partial\!\!\!/ \,-m+i\eta^+\right)\psi(x)}$$

- The functional differentiation of Eq. (152) is straightforward so that

$$\left.\frac{\delta^2 W[\bar{\xi},\xi]}{i\delta\bar{\xi}(x)\,(-i)\delta\xi(y)}\right|_{\bar{\xi}=0=\xi} = i\,S_0(x-y)\,; \tag{153}$$

i.e., the inverse of the Dirac operator, with exactly the Feynman boundary conditions.

- As for scalar quantum field theory, the generating functional for *connected* n-point Green functions is $Z[\bar{\xi},\xi]$, defined via:

$$W[\bar{\xi},\xi] =: \exp\left\{iZ[\bar{\xi},\xi]\right\}. \tag{154}$$

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Fermion Determinant

- What is meant by "$\mathrm{Det}O$," where O is an integral operator?
- Consider a translationally invariant operator

$$O(x,y) = O(x-y) = \int \frac{d^4p}{(2\pi)^4}\, O(p)\, \mathrm{e}^{-i(p,x-y)}. \tag{155}$$

Then, for f as in Eq. (144),

$$\begin{aligned} f[O(x-y)] &= \int \frac{d^4p}{(2\pi)^4} \left\{ f_0 + f_1\, O(p) + f_2\, O^2(p) + [\ldots] \right\} \mathrm{e}^{-i(p,x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}\, f(O(p))\, \mathrm{e}^{-i(p,x-y)}\,. \end{aligned} \tag{156}$$

- Apply this to $\mathcal{N}_0^{\psi} := \mathrm{Det}[iS_0(x-y)]$.

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Fermion Determinant

- Observe: Eq. (141) means one can begin by considering $\mathrm{TrLn}\, iS_0(x-y)$. Writing

$$S_0(p) = m\,\Delta_0(p^2)\left[1+\frac{\not p}{m}\right]\,,\;\Delta_0(p^2) = \frac{1}{p^2-m^2+i\eta^+}\,, \tag{157}$$

$$\Rightarrow S_0(x-y) = \int d^4w\, m\,\Delta_0(x-w)\,\mathcal{F}(w-y)\,, \tag{158}$$

with $\Delta_0(x-y)$ given in Eq. (106) and

$$\mathcal{F}(x-y) = \int \frac{d^4p}{(2\pi)^4}\left[1+\frac{\not p}{m}\right] \mathrm{e}^{-i(p,x-y)}\,. \tag{159}$$

- It follows ($\ln AB = \ln A + \ln B$) that

$$\mathrm{TrLn}\, iS_0(x-y) = \mathrm{Tr}\left\{\mathrm{Ln}\, i\, m\Delta_0(x-y) + \mathrm{Ln}\left[\delta^4(x-y)+\mathcal{F}(x-y)\right]\right\}\,. \tag{160}$$

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Fermion Determinant

- Using Eqs. (142), (156), the second term is

$$\begin{aligned}\mathrm{TrLn}\left[\delta^4(x-y)+\mathcal{F}(x-y)\right] &= \int d^4x \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\ln\left[1+\mathcal{F}(p)\right] \\ &= \int d^4x \int \frac{d^4p}{(2\pi)^4}\,2\,\ln\left[1-\frac{p^2}{m^2}\right] \qquad (161)\end{aligned}$$

- Applying the same equations, the first term is

$$\mathrm{TrLn}\, im\Delta_0(x-y) = \int d^4x \int \frac{d^4p}{(2\pi)^4}\,2\,\ln\left[i\,m\Delta_0(p^2)\right]^2 , \qquad (162)$$

where in both cases $\int d^4x$ measures the (infinite) spacetime volume.

- Combining these results one obtains

$$\boxed{\mathrm{Ln}\,\mathcal{N}_0^{\psi} = \mathrm{TrLn}\, iS_0(x-y) = \int d^4x \int \frac{d^4p}{(2\pi)^4}\,2\,\ln\Delta_0(p^2)}$$

  - The factor $\mathbf{2}$ reflects the spin-degeneracy of the free-fermion's eigenvalues.
  - Including a "colour" degree-of-freedom, this would become "$\boldsymbol{2N_c}$," where N_c is the number of colours.

Gauge Fields

- Generating Function for Gauge Fields (photons, gluons, etc.) can be constructed.
  - Differs from scalar field case because of gauge degree of freedom.
  - One has to carefully implement gauge fixing.
  - That problem is not fully resolved. It involves the so-called *Faddeev-Popov Determinant*, which introduces so-called *ghost fields*.
- We're going to omit this discussion.
  - Details are available on pp. 37-49 in
    http://www.phy.anl.gov/theory/ztfr/LecNotes.pdf
    which also lists many references.
- This omission is not crucial for our development. At this point, the basic ideas of the Functional Integral formulation of Quantum Field Theory are in mind.

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Exercises

- Repeat the derivation of Eq. (93) for $G_c(x_1, x_2, x_3)$.
- Prove Eq. (108).
- Derive Eq. (110).
- Prove Eq. (112).
- Verify Eq. (120).
- Verify Eqs. (122), (123).
- Verify Eqs. (128), (129).
- Verify Eqs. (131).
- Verify Eq. (140).
- Verify Eq. (153).
- Verify Eq. (161).

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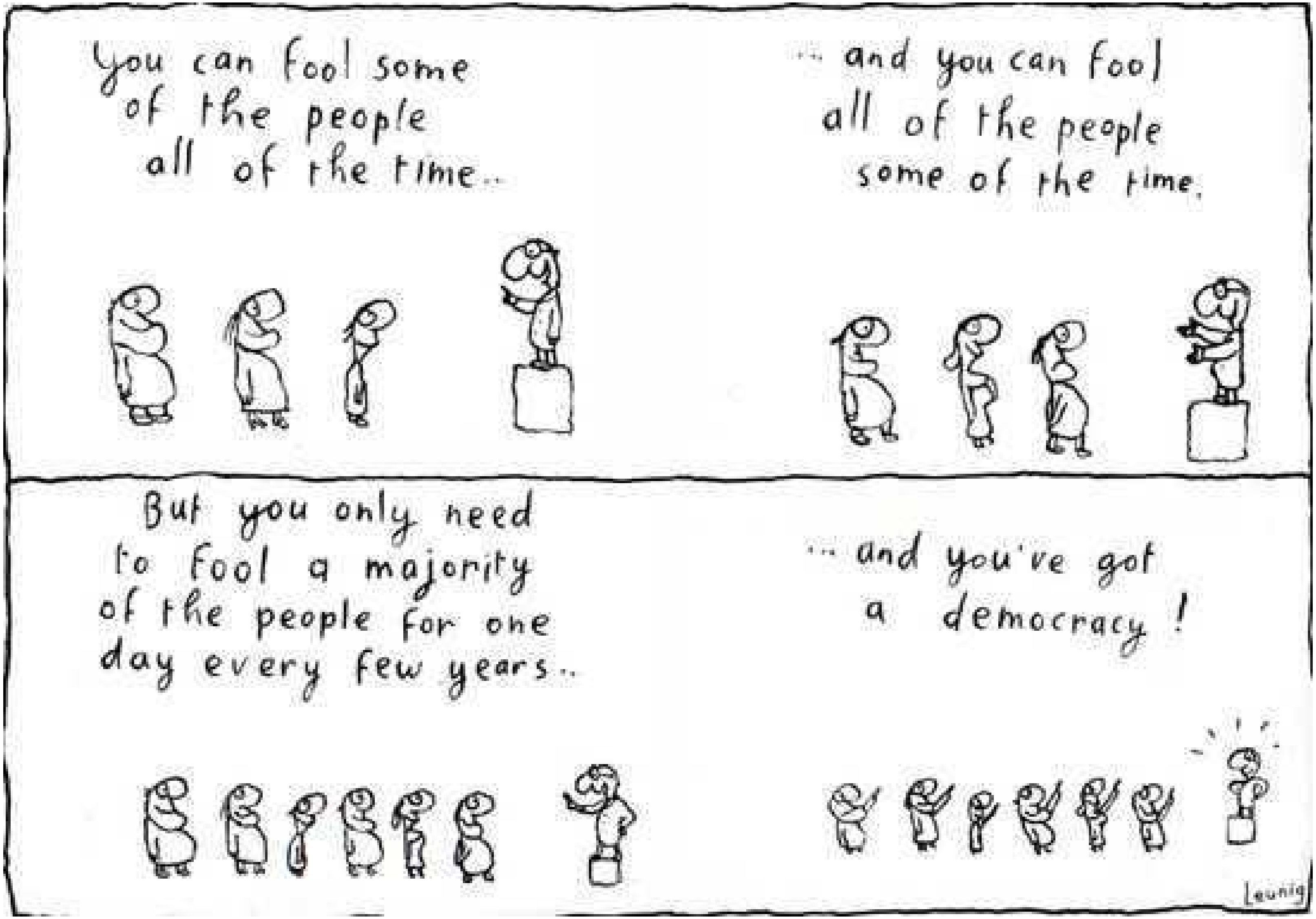
Any Questions?



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Dyson-Schwinger Equations

- It has long been known that, from the field equations of quantum field theory, one can derive a system of coupled integral equations interrelating all of a theory's Green functions:
  - Dyson, F.J. (1949), "The S Matrix In Quantum Electrodynamics," *Phys. Rev*. **75**, 1736.
  - Schwinger, J.S. (1951), "On The Green's Functions Of Quantized Fields: 1 and 2," *Proc. Nat. Acad. Sci*. **37** (1951) 452; *ibid* 455.
- This collection of a countable infinity of equations is called the complex of Dyson-Schwinger equations (DSEs).
- It is an intrinsically nonperturbative complex, which is vitally important in proving the renormalisability of quantum field theories, and at its simplest level the complex provides a generating tool for perturbation theory.
- In the context of quantum electrodynamics (QED) we will illustrate a nonperturbative derivation of two equations in this complex. The derivation of others follows the same pattern.

Photon Vacuum Polarisation

NB. This is one part of the Lamb Shift

- Action for QED with N_f flavours of electomagnetically active fermions:

$$S[A_\mu,\psi,\bar\psi] = \int d^4x \left[\sum_{f=1}^{N_f} \bar\psi^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\lambda_0}\, \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right] . \tag{163}$$

- Manifestly Poincaré covariant action:
  - $\bar\psi^f(x)$, $\psi^f(x)$ are elements of Grassmann algebra that describe the fermion degrees of freedom;
  - m_0^f are the fermions' bare masses and e_0^f, their charges;
  - and $A_\mu(x)$ describes the gauge boson [photon] field, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and λ_0 the bare gauge fixing parameter. (NB. To describe an electron the physical charge $e_f < 0$.)

QED Generating Functional

- The procedure outlined previously can be followed to obtain

$$W[J_\mu,\xi,\bar\xi] = \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]$$
$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4} F^{\mu\nu}(x)\,F_{\mu\nu}(x) - \frac{1}{2\lambda_0}\,\partial^\mu A_\mu(x)\,\partial^\nu A_\nu(x) \right.\right.$$
$$+ \sum_{f=1}^{N_f} \bar\psi^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f$$
$$\left.\left. + J^\mu(x) A_\mu(x) + \bar\xi^f(x)\psi^f(x) + \bar\psi^f(x)\xi^f(x) \right] \right\} , \qquad (164)$$

- simple interaction term: $\boxed{\bar\psi^f\, e_0^f A\!\!\!/ \psi^f}$
- J_μ is an external source for the electromagnetic field
- ξ^f, $\bar\xi^f$ are external sources for the fermion field that, of course, are elements in the Grassmann algebra.

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Functional Field Equations

- Advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu, \bar{\xi}, \xi]$ defined via

$$W[J_\mu, \xi, \bar{\xi}] =: \exp\left\{ iZ[J_\mu, \xi, \bar{\xi}] \right\} \, . \tag{165}$$

- Derivation of a DSE follows simply from observation that the integral of a total derivative vanishes, given appropriate boundary conditions; e.g.,

$$\begin{aligned}
0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\, \frac{\delta}{\delta A_\mu(x)}\, \mathrm{e}^{i\left(S[A_\mu,\psi,\bar{\psi}] + \int d^4x\, \left[\overline{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu\right]\right)} \\
&= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\, \left\{ \frac{\delta S}{\delta A_\mu(x)} + J_\mu(x) \right\} \\
&\quad \times \exp\left\{ i \left(S[A_\mu,\psi,\bar{\psi}] + \int d^4x\, \left[\overline{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu\right] \right) \right\} \\
&= \left\{ \frac{\delta S}{\delta A_\mu(x)} \left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta \bar{\xi}}, -\frac{\delta}{i\delta \xi} \right] + J_\mu(x) \right\} W[J_\mu, \xi, \bar{\xi}] \, ,
\end{aligned} \tag{166}$$

where the last line has meaning as a functional differential operator acting on the generating functional.

Functional Field Equations

- Differentiate Eq. (163) to obtain

$$\frac{\delta S}{\delta A_\mu(x)} = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] A^\nu(x) + \sum_f e_0^f\, \overline{\psi}^f(x)\gamma_\mu \psi^f(x)\,, \quad (167)$$

- Equation (166) then becomes

$$-J_\mu(x) = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] \frac{\delta Z}{\delta J_\nu(x)} + \sum_f e_0^f \left(-\frac{\delta Z}{\delta \xi^f(x)}\gamma_\mu \frac{\delta Z}{\delta \bar{\xi}^f(x)} + \frac{\delta}{\delta \xi^f(x)}\left[\gamma_\mu \frac{\delta\, iZ}{\delta \bar{\xi}^f(x)}\right]\right), \quad (168)$$

where we have divided through by $W[J_\mu, \xi, \bar{\xi}]$.

- Equation (168) represents a compact form of the nonperturbative equivalent of Maxwell's equations.

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One-particle Irreducible Green Function

- Introduce generating functional for one-particle-irreducible (1PI) Green functions: $\Gamma[A_\mu, \psi, \bar{\psi}]$. Obtained from $Z[J_\mu, \xi, \bar{\xi}]$ via a Legendre transformation; namely,

$$Z[J_\mu, \xi, \bar{\xi}] = \Gamma[A_\mu, \psi, \bar{\psi}] + \int d^4x \, \left[\overline{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu\right] . \tag{169}$$

- One-particle-irreducible n-point function or “proper vertex” contains no contributions that become disconnected when a single connected m-point Green function is removed; e.g., via functional differentiation.
  - No diagram representing or contributing to a given proper vertex separates into two disconnected diagrams if only one connected propagator is cut. (Detailed explanation: Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York), pp. 289-294.)

Implications of Legendre Transformation

- A simple generalisation of the analysis beginning on page 43 yields

$$\frac{\delta Z}{\delta J^\mu(x)} = A_\mu(x)\,,\ \frac{\delta Z}{\delta \bar\xi(x)} = \psi(x)\,,\ \frac{\delta Z}{\delta \xi(x)} = -\bar\psi(x)\,, \tag{170}$$

where here the external sources are **nonzero**.

- Hence Γ in Eq. (169) must satisfy

$$\frac{\delta \Gamma}{\delta A^\mu(x)} = -J_\mu(x)\,,\ \frac{\delta \Gamma}{\delta \bar\psi^f(x)} = -\xi^f(x)\,,\ \frac{\delta \Gamma}{\delta \psi^f(x)} = \bar\xi^f(x)\,. \tag{171}$$

(NB. Since the sources are not zero then, e.g.,

$$A_\rho(x) = A_\rho(x;[J_\mu,\xi,\bar\xi]) \;\Rightarrow\; \frac{\delta A_\rho(x)}{\delta J^\mu(y)} \neq 0\,, \tag{172}$$

with analogous statements for the Grassmannian functional derivatives.)

- NB. It is easy to see that setting $\bar\psi = 0 = \psi$ after differentiating Γ gives zero *unless* there are equal numbers of $\bar\psi$ and ψ derivatives. (This is analogous to the result for scalar fields in Eq. (112).)

Green Function's Inverse

- Consider the operator and matrix product (with spinor labels r, s, t)

$$-\int d^4z\, \frac{\delta^2 Z}{\delta\xi_r^f(x)\bar{\xi}_t^h(z)}\, \frac{\delta^2\Gamma}{\delta\psi_t^h(z)\overline{\psi}_s^g(y)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} . \tag{173}$$

- Using Eqs. (170), (171), this simplifies as follows:

$$= \int d^4z\, \frac{\delta\psi_t^h(z)}{\delta\xi_r^f(x)}\, \frac{\delta\xi_s^g(y)}{\delta\psi_t^h(z)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} = \frac{\delta\xi_s^g(y)}{\delta\xi_r^f(x)}\Bigg|_{\psi=\overline{\psi}=0} = \delta_{rs}\,\delta^{fg}\,\delta^4(x-y)\,. \tag{174}$$

- Back in Eq. (168), setting $\bar{\xi}=0=\xi$ one obtains

$$\frac{\delta\Gamma}{\delta A^\mu(x)}\bigg|_{\psi=\overline{\psi}=0} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right] A^\nu(x) - i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu S^f(x,x;[A_\mu])\right] , \tag{175}$$

Identification: $S^f(x,y;[A_\mu]) = -\dfrac{\delta^2 Z}{\delta\xi^f(y)\bar{\xi}^f(x)} = \dfrac{\delta^2 Z}{\delta\bar{\xi}^f(x)\xi^f(y)}$ (no summation on f) , (176)

Green Function's Inverse

- As a direct consequence of Eq. (173) the inverse of this Green function is given by

$$S^f(x,y;[A])^{-1} = \left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(y)}\right|_{\psi=\overline{\psi}=0} . \tag{177}$$

General property: functional derivatives of the generating functional for 1PI Green functions are related to the associated propagator's inverse.

- Clearly the vacuum fermion propagator or connected fermion 2-point function is

$$S^f(x,y) := S^f(x,y;[A_\mu=0])\,. \tag{178}$$

Such vacuum Green functions are keystones in quantum field theory.

- To continue, differentiate Eq. (175) with respect to $A_\nu(y)$ and set $J_\mu(x)=0$:

$$\left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu=0\\ \psi=\overline{\psi}=0}} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y)$$
$$-i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu \frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1}\right] . \tag{179}$$

Inverse of Photon Propagator

- l.h.s. is easily understood – Eqs. (177), (178) define the inverse of the fermion propagator:

$$(D^{-1})^{\mu\nu}(x,y) := \left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu = 0 \\ \psi=\overline{\psi}=0}} . \tag{180}$$

- r.h.s., however, must be simplified and interpreted. First observe that

$$-\frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1} = \tag{181}$$

$$\int d^4u d^4w \left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(w)}\right|_{\psi=\overline{\psi}=0}\right)^{-1} \frac{\delta}{\delta A_\nu(y)}\frac{\delta^2\Gamma}{\delta\psi^f(u)\delta\bar\psi^f(w)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(w)\delta\bar\psi^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1},$$

- Analogue of result for finite dimensional matrices:

$$\begin{aligned}\frac{d}{dx}\left[A(x)A^{-1}(x) = \boldsymbol{I}\right] \quad &= 0 = \quad \frac{dA(x)}{dx}\,A^{-1}(x) + A(x)\,\frac{dA^{-1}(x)}{dx}\\ \Rightarrow \quad & \frac{dA^{-1}(x)}{dx} = -A^{-1}(x)\,\frac{dA(x)}{dx}\,A^{-1}(x)\,.\end{aligned} \tag{182}$$

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Proper Fermion-Photon Vertex

- Equation (181) involves the 1PI 3-point function (no summation on f)

$$e_0^f \Gamma_\mu^f(x,y;z) := \frac{\delta}{\delta A_\nu(z)} \frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \,. \tag{183}$$

  This is the proper fermion-gauge-boson vertex.

- At leading order in perturbation theory

$$\Gamma_\nu^f(x,y;z) = \gamma_\nu \, \delta^4(x-z) \, \delta^4(y-z) \,, \tag{184}$$

  Result can be obtained via explicit calculation of functional derivatives in Eq. (183).

Photon Vacuum Polarisation

- Define the gauge-boson *vacuum polarisation*:

$$\Pi_{\mu\nu}(x,y) = i\sum_f (e_0^f)^2 \int d^4z_1\, d^4z_2\, \mathrm{tr}\left[\gamma_\mu S^f(x,z_1)\Gamma_\nu^f(z_1,z_2;y)S^f(z_2,x)\right], \tag{185}$$

- Gauge-boson vacuum polarisation, or "photon self-energy,"
  - Describes modification of the gauge-boson's propagation characteristics due to the presence of virtual particle-antiparticle pairs in quantum field theory.
  - In particular, the photon vacuum polarisation is an important element in the description of process such as $\rho^0 \to e^+e^-$.
- Eq. (179) may now be expressed as

$$(D^{-1})^{\mu\nu}(x,y) = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y) + \Pi_{\mu\nu}(x,y)\,. \tag{186}$$

- The propagator for a free gauge boson is [use $\Pi_{\mu\nu}(x,y) \equiv 0$ in Eq. (186)]

$$D_0^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2+i\eta])}{q^2+i\eta} - \lambda_0 \frac{q^\mu q^\nu}{(q^2+i\eta)^2}\,, \tag{187}$$

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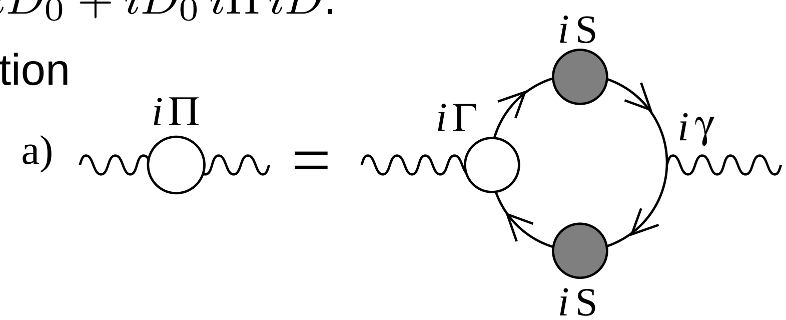
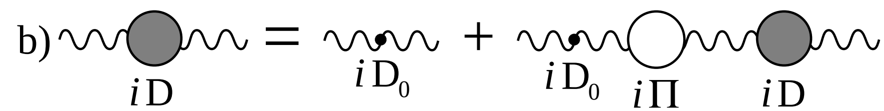
DSE for Photon Propagator

- Then Eq. (186) can be written $iD = iD_0 + iD_0\, i\Pi\, iD$.
  - The is a Dyson-Schwinger Equation



- In presence of interactions;
  i.e., for $\Pi_{\mu\nu} \neq 0$ in Eq. (186),



$$D^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2 + i\eta])}{q^2 + i\eta} \frac{1}{1 + \Pi(q^2)} - \lambda_0 \frac{q^\mu q^\nu}{(q^2 + i\eta)^2}\,, \qquad (188)$$

Used the "Ward-Takahashi identity:" $q_\mu\, \Pi_{\mu\nu}(q) = 0 = \Pi_{\mu\nu}(q)\, q_\nu$,

$$\Rightarrow \Pi^{\mu\nu}(q) = \left(-g^{\mu\nu} q^2 + q^\mu q^\nu\right)\, \Pi(q^2)\,. \qquad (189)$$

- $\Pi(q^2)$ is the polarisation scalar. Independent of the gauge parameter, λ_0, in QED.
  - $\lambda_0 = 1$ is called "Feynman gauge." Useful in perturbative calculations because it simplifies the $\Pi(q^2) = 0$ gauge boson propagator enormously.
  - In nonperturbative applications, however, $\lambda_0 = 0$, "Landau gauge," is most useful because it ensures that the gauge boson propagator is itself transverse.

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Ward-Takahashi Identities

- Ward-Takahashi identities (WTIs) are relations satisfied by n-point Green functions, relations which are an essential consequence of a theory's local gauge invariance; i.e., local current conservation.
- They can be proved directly from the generating functional and have physical implications. For example, Eq. (189) ensures that the photon remains massless in the presence of charged fermions.
- A discussion of WTIs can be found in
  - Bjorken, J.D. and Drell, S.D. (1965), *Relativistic Quantum Fields* (McGraw-Hill, New York), pp. 299-303,
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York), pp. 407-411;
- Their generalisation to non-Abelian theories as "Slavnov-Taylor" identities is described in
  Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin), Chap. 2.

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Vacuum Polarisation in Momentum Space

- In absence of external sources, Eq. (185) can easily be represented in momentum space, because then the 2- and 3-point functions appearing therein must be translationally invariant and hence they can be simply expressed in terms of Fourier amplitudes; i.e., we have

$$i\Pi_{\mu\nu}(q) = -\sum_f (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^{\rm d}} \mathrm{tr}[(i\gamma_\mu)(iS^f(\ell))(i\Gamma^f(\ell,\ell+q))(iS(\ell+q))] \,. \quad (190)$$

  The reduction to a single integral makes momentum space representations most widely used in continuum calculations.

- QED: the vacuum polarisation is directly related to the running coupling constant, which is a connection that makes its importance obvious.
- QCD: connection not so direct but, nevertheless, the polarisation scalar is a key component in the evaluation of the strong running coupling.
- Observed: second derivatives of the generating functional, $\Gamma[A_\mu,\psi,\bar\psi]$, give the inverse-fermion and -photon propagators; third derivative gave the proper photon-fermion vertex. In general, all derivatives of $\Gamma[A_\mu,\psi,\bar\psi]$, higher than two, produce a proper vertex, number and type of derivatives give the number and type of proper Green functions that it can connect.

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Functional Dirac Equation

- Equation (168) is a nonperturbative generalisation of Maxwell's equation in quantum field theory. Its derivation provides the model by which one can obtain an equivalent generalisation of Dirac's equation:

$$
\begin{aligned}
0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\frac{\delta}{\delta\bar\psi^f(x)}\,\mathrm{e}^{i\left(S[A_\mu,\psi,\bar\psi]+\int\, d^4x\,\left[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu\right]\right)} \\
&= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\left\{\frac{\delta S}{\delta\bar\psi^f(x)}+\xi^f(x)\right\} \\
&\quad \times \exp\left\{i\left(S[A_\mu,\psi,\bar\psi]+\int\, d^4x\,\left[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu\right]\right)\right\} \\
&= \left\{\frac{\delta S}{\delta\bar\psi^f(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar\xi},-\frac{\delta}{i\delta\xi}\right]+\eta^f(x)\right\}W[J_\mu,\xi,\bar\xi] \qquad (191)\\
0 &= \left[\xi^f(x)+\left(i\not\partial-m_0^f+e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)\frac{\delta}{i\delta\bar\xi^f(x)}\right]W[J_\mu,\xi,\bar\xi]. \qquad (192)
\end{aligned}
$$

- The last line furnishes a nonperturbative functional equivalent of Dirac's equation.

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Functional Green Function

- Next step ... a functional derivative with respect to ξ^f: $\delta/\delta\xi^f(y)$, yields

$$\delta^4(x-y)W[J_\mu] - \left(i\not\partial - m_0^f + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)}\right) W[J_\mu]\, S^f(x,y;[A_\mu]) = 0\,, \quad (193)$$

  after setting $\xi^f = 0 = \bar\xi^f$, where $W[J_\mu] := W[J_\mu,0,0]$ and $S(x,y;[A_\mu])$ is defined in Eq. (176).

- Now, using Eqs. (165), (171), this can be rewritten

$$\delta^4(x-y) - \left(i\not\partial - m_0^f + e_0^f \not A(x;[J]) + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)}\right) S^f(x,y;[A_\mu]) = 0\,, \quad (194)$$

  which defines the nonperturbative connected 2-point fermion Green function

  - NB. This is clearly the functional equivalent of Eq. (69):

$$[i\not\partial_{x'} - e\not A(x') - m]\, S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \quad (195)$$

    namely, Differential Operator Green Function for the Interacting Dirac Theory.

DSE for Fermion Propagator

- The electromagentic four-potential vanishes in the absence of an external source; i.e., $A_\mu(x;[J=0])=0$
- Remains only to exhibit the content of the remaining functional differentiation in Eq. (194), which can be accomplished using Eq. (181):

$$\frac{\delta}{i\delta J^\mu(x)}\, S^f(x,y;[A_\mu]) = \int d^4z\, \frac{\delta A_\nu(z)}{i\delta J^\mu(x)}\, \frac{\delta}{\delta A_\nu(z)} \left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(y)}\right|_{\psi=\overline{\psi}=0}\right)^{-1}$$

$$\begin{aligned} &= -e_0^f \int d^4z\, d^4u\, d^4w\, \frac{\delta A_\nu(z)}{i\delta J_\mu(x)}\, S^f(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y) \\ &= -e_0^f \int d^4z\, d^4u\, d^4w\, iD_{\mu\nu}(x-z)\, S^f(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y)\,, \end{aligned} \tag{196}$$

In the last line, we have set $J=0$ and used Eq. (180).

- Hence in the absence of external sources Eq. (194) is equivalent to

$$\begin{aligned} \delta^4(x-y) &= \left(i\not\partial - m_0^f\right) S^f(x,y) \\ &- i\,(e_0^f)^2 \int d^4z\, d^4u\, d^4w\, D^{\mu\nu}(x,z)\, \gamma_\mu\, S(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y)\,. \end{aligned} \tag{197}$$

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Fermion Self Energy

- Photon vacuum polarisation was introduced to re-express the DSE for the gauge boson propagator, Eq. (185). Analogue, one can define a fermion self-energy:

$$\Sigma^f(x,z) = i(e_0^f)^2 \int d^4u\, d^4w\; D^{\mu\nu}(x,z)\, \gamma_\mu\, S(x,u)\, \Gamma_\nu(u,w;z)\,, \tag{198}$$

so that Eq. (197) assumes the form

$$\int d^4z\, \left[\left(i\partial\!\!\!/_x - m_0^f \right) \delta^4(x-z) - \Sigma^f(x,z) \right] S(z,y) = \delta^4(x-y)\,. \tag{199}$$

- Using property that Green functions are translationally invariant in the absence of external sources:

$$-i\Sigma^f(p) = (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4}\, [iD^{\mu\nu}(p-\ell)]\, [i\gamma_\mu]\, [iS^f(\ell)]\, [i\Gamma_\nu^f(\ell,p)]\,. \tag{200}$$

Now follows from Eq. (199) that connected fermion 2-point function in momentum space is

$$S^f(p) = \frac{1}{p\!\!\!/ - m_0^f - \Sigma^f(p) + i\eta^+}\,. \tag{201}$$

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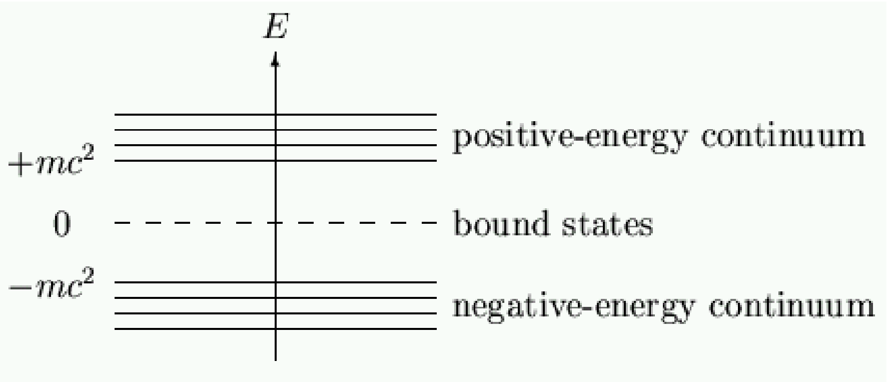
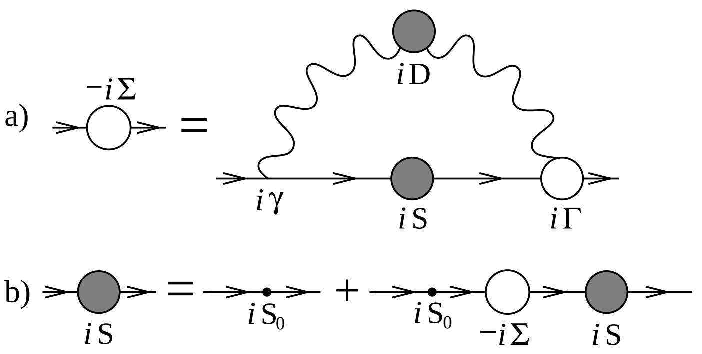
Gap Equation

- Equation (200) is the **exact** *Gap Equation*.



- Describes manner in which propagation characteristics of a fermion moving through ground state of QED (the QED vacuum) is altered by the repeated emission and reabsorption of virtual photons.



- Equation can also describe the real process of Bremsstrahlung. Furthermore, solution of analogous equation in QCD provides information about dynamical chiral symmetry breaking and also quark confinement.

Perturbative Calculation of Gap

- Keystone of strong interaction physics is dynamical chiral symmetry breaking (DCSB). In order to understand DCSB one must first come to terms with explicit chiral symmetry breaking. Consider then the DSE for the quark self-energy in QCD:

$$-i\,\Sigma(p) = -g_0^2 \int \frac{d^4\ell}{(2\pi)^4}\, D^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a \gamma_\mu\, S(\ell)\, i\Gamma^a_\nu(\ell,p)\,, \qquad (202)$$

  where the flavour label is suppressed.

- Form is precisely the same as that in QED, Eq. (200) but . . .
  - colour (Gell-Mann) matrices: $\{\lambda^a; a = 1,\ldots,8\}$ at the fermion-gauge-boson vertex
  - $D^{\mu\nu}(\ell)$ is the connected gluon 2-point function
  - $\Gamma^a_\nu(\ell,\ell')$ is the proper quark-gluon vertex

- One-loop contribution to quark's self-energy obtained by evaluating r.h.s. of Eq. (202) using the free quark and gluon propagators, and the quark-gluon vertex:

$$\Gamma^{a\,(0)}_\nu(\ell,\ell') = \frac{1}{2}\lambda^a \gamma_\nu\,. \qquad (203)$$

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Explicit Leading-Order Calculation

- $$- i\,\Sigma^{(2)}(p) = -g_0^2 \int \frac{d^4k}{(2\pi)^4} \left(-g^{\mu\nu} + (1-\lambda_0)\frac{k^\mu k^\nu}{k^2+i\eta^+}\right) \frac{1}{k^2+i\eta^+} \times \frac{i}{2}\lambda^a\gamma_\mu \frac{1}{\not k + \not p - m_0 + i\eta^+} \frac{i}{2}\lambda^a\gamma_\mu\,. \tag{204}$$

- To proceed, first observe that Eq. (204) can be re-expressed as

$$-i\,\Sigma^{(2)}(p) = -g_0^2\, C_2(R) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k+p)^2 - m_0^2 + i\eta^+} \frac{1}{k^2+i\eta^+} \times \left\{ \gamma^\mu\,(\not k + \not p + m_0)\,\gamma_\mu - (1-\lambda_0)\,(\not k - \not p + m_0) - 2\,(1-\lambda_0)\,\frac{(k,p)\,\not k}{k^2+i\eta^+} \right\}, \tag{205}$$

where we have used $\boxed{\frac{1}{2}\lambda^a\,\frac{1}{2}\lambda^a = C_2(R)\,\boldsymbol{I}_c\,;\; C_2(R) = \frac{N_c^2-1}{2N_c}}$, with N_c the number of colours ($N_c = 3$ in QCD), and $\boldsymbol{I}_c$ is the identity matrix in colour space.

Explicit Leading-Order Calculation

- Now note that $2\,(k,p) = [(k+p)^2 - m_0^2] - [k^2] - [p^2 - m_0^2]$ and hence

$$
\begin{aligned}
-\,i\,\Sigma^{(2)}(p) \quad = \quad & -g_0^2\, C_2(R) \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\, \frac{1}{k^2 + i\eta^+} \\
& \Big\{ \gamma^\mu\, (\not k + \not p + m_0)\, \gamma_\mu + (1-\lambda_0)\, (\not p - m_0) \\
& + (1-\lambda_0)\, (p^2 - m_0^2)\, \frac{\not k}{k^2 + i\eta^+} \\
& - (1-\lambda_0)\, [(k+p)^2 - m_0^2]\, \frac{\not k}{k^2 + i\eta^+} \Big\}\,. \qquad (206)
\end{aligned}
$$

- Focus on the last term:

$$
\begin{aligned}
& \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\, \frac{1}{k^2 + i\eta^+}\, [(k+p)^2 - m_0^2]\, \frac{\not k}{k^2 + i\eta^+} \\
= \quad & \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 + i\eta^+}\, \frac{\not k}{k^2 + i\eta^+} = 0 \qquad (207)
\end{aligned}
$$

because the integrand is odd under $k \to -k$, and so this term in Eq. (206) vanishes.

Explicit Leading-Order Calculation

- $$-i\,\Sigma^{(2)}(p) = -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}$$
$$\left\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not p-m_0)+(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not k}{k^2+i\eta^+}\right\}.$$

- Consider the second term:
$$(1-\lambda_0)\,(\not p-m_0)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\,.$$

In particular, focus on the behaviour of the integrand at large k^2:
$$\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\overset{k^2\to\pm\infty}{\sim}\frac{1}{(k^2-m_0^2+i\eta^+)\,(k^2+i\eta^+)}\,. \tag{208}$$

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Wick Rotation

- Integrand has poles in the second and fourth quadrant of the complex-k_0-plane but vanishes on any circle of radius $R \to \infty$ in this plane. That means one may rotate the contour anticlockwise to find

$$\int_0^\infty dk^0 \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)}$$
$$= \int_0^{i\infty} dk^0 \, \frac{1}{([k^0]^2 - \vec{k}^2 - m_0^2 + i\eta^+)([k^0]^2 - \vec{k}^2 + i\eta^+)}$$
$$\stackrel{k^0 \to i k_4}{=} \quad i \int_0^\infty dk_4 \, \frac{1}{(-k_4^2 - \vec{k}^2 - m_0^2)\,(-k_4^2 - \vec{k}^2)} \,. \tag{209}$$

- Performing a similar analysis of the $\int_{-\infty}^0$ part, one obtains the complete result:

$$\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)}$$
$$= i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \, \frac{1}{(-\vec{k}^2 - k_4^2 - m_0^2)\,(-\vec{k}^2 - k_4^2)} \,. \tag{210}$$

These two steps constitute what is called a *Wick rotation*.

Euclidean Integral

- The integral on the r.h.s. is defined in a four-dimensional Euclidean space; i.e., $k^2 := k_1^2 + k_2^2 + k_3^2 + k_4^2 \geq 0$, with k^2 nonnegative.
- A general vector in this space can be written in the form:

$$(k) = |k| \left(\cos\phi\, \sin\theta\, \sin\beta, \sin\phi\, \sin\theta\, \sin\beta, \cos\theta\, \sin\beta, \cos\beta\right); \tag{211}$$

i.e., using hyperspherical coordinates, and clearly $k^2 = |k|^2$.

- In this Euclidean space using these coordinates the four-vector measure factor is

$$\begin{aligned} & \int d_E^4 k\, f(k_1, \ldots, k_4) \\ = \quad & \frac{1}{2} \int_0^\infty dk^2\, k^2 \int_0^\pi d\beta\, sin^2\beta \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, f(k, \beta, \theta, \phi)\,. \end{aligned} \tag{212}$$

Euclidean Integral

- Returning to Eq. (208) and making use of the material just introduced, the large k^2 behaviour of the integral can be determined via

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k+p)^2 - m_0^2 + i\eta^+} \, \frac{1}{k^2 + i\eta^+} & \\ \approx \quad \frac{i}{16\pi^2} \int_0^\infty dk^2 \frac{1}{(k^2 + m_0^2)} & \\ = \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} dx \, \frac{1}{x + m_0^2} & \\ = \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \ln\left(1 + \Lambda^2/m_0^2\right) \to \infty \, ; & \end{aligned} \tag{213}$$

- After all this work, the result is meaningless: the one-loop contribution to the quark's self-energy is divergent!

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Regularisation and Renormalisation

- Such “ultraviolet” divergences, and others which are more complicated, arise whenever loops appear in perturbation theory. (The others include “infrared” divergences associated with the gluons’ masslessness; e.g., consider what would happen in Eq. (213) with $m_0 \to 0$.)
- In a *renormalisable* quantum field theory there exists a well-defined set of rules that can be used to render perturbation theory sensible.
  - First, however, one must *regularise* the theory; i.e., introduce a cutoff, or use some other means, to make finite every integral that appears. Then each step in the calculation of an observable is rigorously sensible.
  - Renormalisation follows; i.e, the absorption of divergences, and the redefinition of couplings and masses, so that finally one arrives at $\mathcal{S}$-matrix amplitudes that are finite and physically meaningful.
- The regularisation procedure must preserve the Ward-Takahashi identities (the Slavnov-Taylor identities in QCD) because they are crucial in proving that a theory can be sensibly renormalised.
- A theory is called renormalisable if, and only if, number of different types of divergent integral is finite. Then only finite number of masses & couplings need to be renormalised; i.e., *a priori* the theory has only a finite number of undetermined parameters that must be fixed through comparison with experiments.

Renormalised One-Loop Result

- Don't have time to explain and illustrate the procedure. Interested? Read . . . Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin).
- Answer, in Momentum Subtraction Scheme:
$\Sigma^{(2)}_{R}(\not p) = \Sigma^{(2)}_{VR}(p^2)\,\not p + \Sigma^{(2)}_{SR}(p^2)\,\mathbf{1}_D$;

$$\Sigma^{(2)}_{VR}(p^2;\zeta^2) = \frac{\alpha(\zeta)}{\pi}\,\lambda(\zeta)\,\frac{1}{4}\,C_2(R)\,\Bigg\{ -m^2(\zeta)\left(\frac{1}{p^2}+\frac{1}{\zeta^2}\right) + \left(1-\frac{m^4(\zeta)}{p^4}\right)\ln\left(1-\frac{p^2}{m(\zeta)^2}\right) - \left(1-\frac{m^4(\zeta)}{\zeta^4}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\Bigg\}\,,$$

$$\Sigma^{(2)}_{SR}(p^2;\zeta^2) = m(\zeta)\,\frac{\alpha(\zeta)}{\pi}\,\frac{1}{4}\,C_2(R)\,\Bigg\{ -\left[3+\lambda(\zeta)\right] \times\left[\left(1-\frac{m^2(\zeta)}{p^2}\right)\ln\left(1-\frac{p^2}{m^2(\zeta)}\right) - \left(1+\frac{m^2(\zeta)}{\zeta^2}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\right]\Bigg\}\,,$$

where the renormalised quantities depend on the point at which the renormalisation has been conducted;
e.g., $\alpha(\zeta)$ is the running coupling, $m(\zeta)$ is the running quark mass.

Observations on Quark Self Energy

- QCD is Asymptotically Free. Hence, at some large spacelike $p^2 = \zeta^2$ the propagator is exactly the free propagator *except* that the bare mass is replaced by the renormalised mass.
- At one-loop order, the vector part of the dressed self energy is proportional to the running gauge parameter. In Landau gauge, that parameter is zero. Hence, the vector part of the renormalised dressed self energy vanishes at one-loop order in perturbation theory.
- The scalar part of the dressed self energy is proportional to the renormalised current-quark mass.
  - This is true at one-loop order, and at every order in perturbation theory.
  - Hence, if current-quark mass vanishes, then $\Sigma_{SR} \equiv 0$ in perturbation theory. That means if one starts with a chirally symmetric theory, one ends up with a chirally symmetric theory: NO DCSB in perturbation theory.

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Exercises

- Verify Eq. (171).
- Verify Eq. (181).
- Verify Eq. (190).
- Verify Eq. (194).
- Verify Eq. (213).

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