Formula for calculating the $\pi N \to \pi\pi N$ cross sections
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In the center of mass frame, the momentum variables of the $\pi N \to \pi\pi N$ reaction with invariant mass W can be specified as

$$a(\vec{p}_a) + b(\vec{p}_b) \to c(\vec{p}_c) + d(\vec{p}_d) + e(\vec{p}_e) \ , \tag{1}$$

where $\vec{p}_a = -\vec{p}_b = \vec{k}$ with k defined by $W = E_a(k)+E_b(k)$, $\vec{p}_c+\vec{p}_d = -\vec{p}_e = \vec{k}'$, and $(c+d+e)$ can be any possible charged states formed from two pions and one nucleon. The total cross section of the process Eq. (1) can be written as

$$\sigma^{rec}_{ab\to cde} = \int_{m_c+m_d}^{W-m_e} \frac{d\sigma^{rec}}{dM_{cd}} dM_{cd} \ , \tag{2}$$

with

$$\frac{d\sigma^{rec}}{dM_{cd}} = \frac{\rho_i}{k^2} 16\pi^3 \int d\Omega_{k_{cd}} d\Omega_{k'} \frac{k_{cd}k'}{W} \frac{1}{(2s_a+1)(2s_b+1)} \sum_{i,f} |\sqrt{E_c E_d E_e} \langle \vec{p}_c \vec{p}_d \vec{p}_e, f|T|\vec{k}, i\rangle|^2 \ , \tag{3}$$

where $\rho_i = \pi\frac{kE_a(k)E_b(k)}{W}$,$i, f$ denote all spin (s_a, s_{az}) and isospin (t_a, t_{az}) quantum numbers, and $\sum_{i,f}$ means summing over only spin quantum numbers. For a given invariant mass M_{cd}, $\vec{k}_{cd}$ is the relative momentum between c and d in the center of mass of the sub-system (cd) . It follows that k' and k_{cd} are defined by W and M_{cd}:

$$\begin{aligned} M_{cd} &= E_c(k_{cd}) + E_d(k_{cd}) \, , \\ W &= E_e(k') + E_{cd}(k') \, , \\ E_{cd}(k') &= \sqrt{M^2_{cd} + (k')^2} \, . \end{aligned} \tag{4}$$

(5)

The T-matrix elements in the Eq. (3) are of the following form

$$\begin{aligned} \langle \vec{p}_c \vec{p}_d \vec{p}_e, f|T|\vec{k}, i\rangle &= \sum_{s_{R_z}, t_{R_z}} \frac{\langle \vec{p}_c, s_{cz}, t_{cz}; \vec{p}_d, s_{dz}, t_{dz}|H_I|\vec{k}', s_{Rz}, t_{R_z}\rangle}{W - E_e(k') - E_R(k') - \Sigma_{eR}(k', E)} \\ &\quad \times \langle \vec{k}', s_{Rz}, t_{Rz}; -\vec{k}', s_{ez}, t_{ez}|T|\vec{k}, s_{az}, t_{az}; -\vec{k}, s_{bz}, t_{bz}\rangle \ , \end{aligned} \tag{6}$$

where R is a bare state which has $R \to cd$ decay channel. For the $eR = \pi\Delta$ and $eR = \rho N$ channels, the self-energies are explicitly given by

$$\Sigma_{\pi\Delta}(k;W) = \frac{m_\Delta}{E_\Delta(k)} \int q^2 dq \frac{M_{\pi N}(q)}{[M^2_{\pi N}(q)+k^2]^{1/2}} \frac{|f_{\Delta\to\pi N}(q)|^2}{W - E_\pi(k) - [M^2_{\pi N}(q)+k^2]^{1/2} + i\epsilon}, \tag{7}$$

$$\Sigma_{\rho N}(k;W) = \frac{m_\rho}{E_\rho(k)} \int q^2 dq \frac{M_{\pi\pi}(q)}{[M^2_{\pi\pi}(q)+k^2]^{1/2}} \frac{|f_{\rho\to\pi\pi}(q)|^2}{W - E_N(k) - [M^2_{\pi\pi}(q)+k^2]^{1/2} + i\epsilon}, \tag{8}$$

where $m_\Delta = 1280$ MeV, $m_\rho = 812$ MeV, $M_{\pi N}(q) = E_\pi(q) + E_N(q)$, and $M_{\pi\pi}(q) = E_\pi(q) + E_\pi(q)$. The form factors $f_{\Delta\to\pi N}(q)$ and $f_{\rho\to\pi\pi}(q)$ are for describing the $\Delta \to \pi N$ and $\rho \to \pi\pi$ decays in the Δ and ρ rest frames, respectively. They are parametrized as:

$$
\begin{aligned}
f_{\Delta\to\pi N}(q) &= -i\frac{(0.98)}{[2(m_N + m_\pi)]^{1/2}}\left(\frac{q}{m_\pi}\right)\left(\frac{1}{1+[q/(358\ \text{MeV})]^2}\right)^2, && (9)\\
f_{\rho\to\pi\pi}(q) &= \frac{(0.6684)}{\sqrt{m_\pi}}\left(\frac{q}{(461\ \text{MeV})}\right)\left(\frac{1}{1+[q/(461\ \text{MeV})]^2}\right)^2. && (10)
\end{aligned}
$$

The σ self-energy $\Sigma_{\sigma N}(k;E)$ is calculated from a $\pi\pi$ s-wave scattering model with a vertex function $g(q)$ for the $\sigma \to \pi\pi$ decay and a separable interaction $v(q',q) = h_0 h(q')h(q)$. The resulting form is

$$
\Sigma_{\sigma N}(k;W) = \langle gG_{\pi\pi}g\rangle(k;W) + \tau(k;E)[\langle gG_{\pi\pi}h\rangle(k;W)]^2, \qquad (11)
$$

with

$$
\begin{aligned}
\tau(k;W) &= \frac{h_0}{1 - h_0\langle hG_{\pi\pi}h\rangle(k;W)}, && (12)\\
\langle hG_{\pi\pi}h\rangle(k;W) &= \int dq q^2 \frac{M_{\pi\pi}(q)}{[M^2_{\pi\pi}(q)+k^2]^{1/2}} \\
&\qquad \times \frac{h(q)^2}{W - E_N(k) - [M^2_{\pi\pi}(q)+k^2]^{1/2} + i\varepsilon}, && (13)\\
\langle gG_{\pi\pi}g\rangle(k;W) &= \frac{m_\sigma}{E_\sigma(k)}\int dq q^2 \frac{M_{\pi\pi}(q)}{[M^2_{\pi\pi}(q)+k^2]^{1/2}} \\
&\qquad \times \frac{g(q)^2}{W - E_N(k) - [M^2_{\pi\pi}(q)+k^2]^{1/2} + i\varepsilon}, && (14)\\
\langle gG_{\pi\pi}h\rangle(k;W) &= \sqrt{\frac{m_\sigma}{E_\sigma(k)}}\int dq q^2 \frac{M_{\pi\pi}(q)}{[M^2_{\pi\pi}(q)+k^2]^{1/2}} \\
&\qquad \times \frac{g(q)h(q)}{W - E_N(k) - [M^2_{\pi\pi}(q)+k^2]^{1/2} + i\varepsilon}. && (15)
\end{aligned}
$$

In the above equations, $m_\sigma = 700.0$ MeV and the form factors are

$$
\begin{aligned}
g(p) &= \frac{g_0}{\sqrt{m_\pi}}\frac{1}{1+(cp)^2}, && (16)\\
h(p) &= \frac{1}{m_\pi}\frac{1}{1+(dp)^2}. && (17)
\end{aligned}
$$

where $g_0 = 1.638$, $h_0 = 0.556$, $c = 1.02$ fm, and $d = 0.514$ fm.

For any spins and isospins and c.m. momenta $\vec{p}$ and $\vec{p}\,'$, the $MB \to M'B'$ T-matrix elements in Eq.(6) are in general defined by

$$
\begin{aligned}
&\langle \vec{p}\,', s_{M'z}, t_{M'z}; -\vec{p}\,', s_{B'z}, t_{B'z}|T|\vec{p}, s_{Mz}, t_{Mz}; -\vec{p}, s_{Bz}, t_{Bz}\rangle \\
&= \sum_{JM,TT_z}\ \sum_{L'M'_L,S'S'_z}\ \sum_{LM_L,SS_z} Y_{L',M'_L}(\hat{p}')Y^*_{L,M_L}(\hat{p}) \\
&\times\langle s_{M'}, s_{B'}, s_{M'z}, s_{B'z}|S', S'_z\rangle\langle L', S', M'_L, S'_z|J, M\rangle\langle t_{M'}, t_{B'}, t_{M'z}, t_{B'z}|T, T_z\rangle \\
&\times\langle s_M, s_B, s_{Mz}, s_{Bz}|S, S_z\rangle\langle L, S, M_L, S_z|J, M\rangle\langle t_M, t_B, t_{Mz}, t_{Bz}|T, T_z\rangle \\
&\times t^{JT}_{L'S'M'B',LSMB}(p', p, W), \qquad (18)
\end{aligned}
$$

where the matrix elements $t^{JT}_{L'S'M'B',LS\pi N}(p',p,W)$ for $M'B'=\pi\Delta,\sigma N,\rho N$ are the PWA from the ANL-Osaka model.

The matrix elements of H_I of Eq. (6) describe the decay of a resonance $R=\Delta,\rho,\sigma$ into a two-particle state cd. It is of the following expression

$$\begin{aligned}
&\langle\vec{p}_c,s_{cz},t_{cz};\vec{p}_d,s_{dz},t_{dz}|H_I|\vec{k}',s_{Rz},t_{Rz}\rangle\\
&= \delta(\vec{p}_c+\vec{p}_d-\vec{k}')\sqrt{\frac{E_c(k_{cd})E_d(k_{cd})M_R}{E_c(p_c)E_d(p_d)E_R(k')}}\langle\vec{k}_{cd},s_{cz},t_{cz};-\vec{k}_{cd},s_{dz},t_{dz}|H_I|\vec{0},s_{Rz},t_{Rz}\rangle, \quad (19)
\end{aligned}$$

with

$$\begin{aligned}
&\langle\vec{k}_{cd},s_{cz},t_{cz};-\vec{k}_{cd},s_{dz},t_{dz}|H_I|\vec{0},s_{Rz},t_{Rz}\rangle\\
&= \sum_{L_{cd},S_{cd},m_{cd},S_{cdz}}[\langle s_c,s_d,s_{cz},s_{dz}|S_{cd},S_{cdz}\rangle\langle L_{cd},S_{cd},m_{cd},S_{cdz}|s_R,s_{Rz}\rangle\\
&\quad\times\langle t_c,t_d,t_{cz},t_{dz}|t_R,t_{Rz}\rangle Y_{L_{cd},m_{cd}}(\hat{k}_{cd})F_{L^R_{cd},S^R_{cd}}(k_{cd})]\delta_{L_{cd},L^R_{cd}}\delta_{S_{cd},S^R_{cd}}\ . \quad (20)
\end{aligned}$$

The vertex functions are

$$F_{L^{\Delta}_{\pi N},S^{\Delta}_{\pi N}}(q) = if_{\Delta\to\pi N}(q)\ , \quad (21)$$

$$F_{L^{\sigma}_{\pi\pi},S^{\sigma}_{\pi\pi}}(q) = \sqrt{2}g(q)\ , \quad (22)$$

$$F_{L^{\rho}_{\pi\pi},S^{\rho}_{\pi\pi}}(q) = (-1)\sqrt{2}f_{\rho\to\pi\pi}(q)\ , \quad (23)$$

where $L^{\Delta}_{\pi N}=1, S^{\Delta}_{\pi N}=3/2$, $L^{\sigma}_{\pi\pi}=0, S^{\sigma}_{\pi\pi}=0$, $L^{\rho}_{\pi\pi}=1, S^{\rho}_{\pi\pi}=1$. Here it is noted that the factor $\sqrt{2}$ in Eqs. (22)-(23) comes from the Bose symmetry of pions, and the phase factor i and (-1) are chosen to be consistent with the non-resonant interactions involving $\pi N\Delta$, $\sigma\pi\pi$ and $\rho\pi\pi$ vertex interactions. The form factors $f_{\Delta\to\pi N}(q)$ and $f_{\rho\to\pi\pi}(q)$ have been in Eqs.(9)-(10) and $g(q)$ in Eq.(16).

With the above equations, the contribution from $\pi N\to\pi\Delta\to\pi\pi N$ to the total cross section $\sigma^{rec}_{\pi N\to\pi\pi N}$, as defined by Eq. (2)-(3), can be written as

$$\sigma^{rec}_{\pi\Delta}(W) = \int^{W-m_\pi}_{m_N+m_\pi}dM_{\pi N}\frac{M_{\pi N}}{E_\Delta(k)}\frac{\Gamma_\Delta/(2\pi)}{|W-E_\pi(k)-E_\Delta(k)-\Sigma_{\pi\Delta}(k,W)|^2}\times\sigma_{\pi N\to\pi\Delta}\ , \quad (24)$$

where k and $E_\Delta(k)$ are defined by W and $M_{\pi N}$

$$k = \frac{1}{2W}[(W^2-M^2_{\pi N}-m^2_\pi)^2-4M^2_{\pi N}m^2_\pi]^{1/2}\ , \quad (25)$$

$$E_\Delta(k) = [m^2_\Delta+k^2]^{1/2}\ , \quad (26)$$

$\Sigma_{\pi\Delta}(k,W)$is defined in Eq. (7), $\Gamma_\Delta=-2Im[\Sigma_{\pi\Delta}(k=0,W)]$, and

$$\begin{aligned}
\sigma_{\pi N\to\pi\Delta} &= \frac{4\pi}{k_0^2}\sum_{JT,L'S',LS}\frac{2J+1}{(2S_N+1)(2S_\pi+1)}|\rho^{1/2}_{\pi\Delta}(k)t^{JT}_{L'S'\pi\Delta,LS\pi N}(k,k_0;W)\rho^{1/2}_{\pi N}(k_0)|^2\\
&\quad\times\langle t_\pi,t_N,t^z_\pi,t^z_N|T,T^z\rangle^2\ , \quad (27)
\end{aligned}$$

where k_0 is defined by $W=E_\pi(k_0)+E_N(k_0)$ and $\rho_{ab}(k)=\pi kE_a(k)E_b(k)/W$. Similarly, the contributions of $\pi N\to\rho N\to\pi\pi N$ and $\pi N\to\sigma N\to\pi\pi N$ to the total cross section $\sigma^{rec}_{\pi N\to\pi\pi N}$ are

$$\sigma^{rec}_{aN}(W) = \int^{W-m_N}_{2m_\pi}dM_{\pi\pi}\frac{M_{\pi\pi}}{E_a(k)}\frac{\Gamma_a/(2\pi)}{|W-E_N(k)-E_a(k)-\Sigma_{aN}(k,W)|^2}\times\sigma_{\pi N\to aN}\ , \quad (28)$$

where $a=\rho,\sigma$, k is defined by $M_{\pi\pi}$ and W

$$k \ = \ \frac{1}{2W}[(W^2-M_{\pi\pi}^2-m_N^2)^2-4M_{\pi\pi}^2m_N^2]^{1/2}\,, \tag{29}$$

$$E_a(k) \ = \ [m_a^2+k^2]^{1/2}\,, \tag{30}$$

$\Sigma_{aN}(k,W)$ for $aN=\rho N,\sigma N$ are is defined in Eqs.(104) and (107), $\Gamma_a=-2Im[\Sigma_{aN}(k=0,W)]$, and

$$\begin{aligned}\sigma_{\pi N\to aN} \ &= \ \frac{4\pi}{k_0^2}\sum_{JT,L'S',LS}\frac{2J+1}{(2S_N+1)(2S_\pi+1)}|\rho_{aN}^{1/2}(k)t^{JT}_{L'S'aN,LS\pi N}(k,k_0;W)\rho_{\pi N}^{1/2}(k_0)|^2\\ &\times\langle t_\pi,t_N,t_\pi^z,t_N^z|T,T^z\rangle^2\ .\end{aligned} \tag{31}$$

To perform calculations, we need to have the partial-wave amplitudes $t^{JT}_{L'S'M'B',LS\pi N}(p,k,W)$ for $M'B'=\pi\Delta,\rho N,\sigma N$. These PWA from ANL-Osaka model can be obtained from the webpage which present the following :

$$\begin{aligned}&<\pi\Delta|T(W)|\pi N>=-\rho_{\pi\Delta}^{1/2}(p_\Delta)t^{JT}_{L'S'\pi\Delta,LS\pi N}(p_\Delta,k,W)\rho^{1/2}_{\pi N}(k)\ ,\\ &<\rho N|T(W)|\pi N>=-\rho_{\rho N}^{1/2}(p_\rho)t^{JT}_{L'S'\rho N,LS\pi N}(p_\rho,k,W)\rho^{1/2}_{\pi N}(k)\ ,\\ &<\sigma N|T(W)|\pi N>=-\rho_{\sigma N}^{1/2}(p_\sigma)t^{JT}_{L'S'\sigma N,LS\pi N}(p_\sigma,k,W)\rho^{1/2}_{\pi N}(k)\ ,\end{aligned} \tag{32}$$

Here the phase space factors account for the effects due to $\Delta\to\pi N$, $\sigma\to\pi\pi$ and $\rho\to\pi\pi$ decays. Explicitly, we have

$$\rho_{\pi\Delta}(p_\Delta) \ = \ \pi\frac{p_\Delta E_\Delta(p_\Delta)E_\pi(p_\Delta)}{W}\ , \tag{33}$$

where p_Δ and $E_\Delta(p_\Delta)$ are defined by W and the invariant mass $M_{\pi N}$ in the integrations of Eqs.(2) and (24)

$$p_\Delta \ = \ \frac{1}{2W}[(W^2-M_{\pi N}^2-m_\pi^2)^2-4M_{\pi N}^2m_\pi^2]^{1/2}\,, \tag{34}$$

$$E_\Delta(p_\Delta) \ = \ [M_{\pi N}^2+p_\Delta^2]^{1/2}\,, \tag{35}$$

For the calculations of Eqs.(2) and (24), we thus present $<\pi\Delta|T(W)|\pi N>$ in the range $0\leq p_\Delta\leq p_{\Delta,max}$ with

$$p_{\Delta,max}=\frac{1}{2W}[(W^2-(m_\pi+m_N)^2-m_\pi^2)^2-4(m_\pi+m_N)^2m_\pi^2]^{1/2}\ . \tag{36}$$

For the ρN and σN channels, we have

$$\rho_{\sigma N}^{1/2}(p_\sigma) \ = \ \pi\frac{p_\sigma E_\sigma(p_\sigma)E_N(p_\sigma)}{W}\ , \tag{37}$$

$$\rho_{\rho N}^{1/2}(p_\rho) \ = \ \pi\frac{p_\rho E_\rho(p_\rho)E_N(p_\rho)}{W}\ , \tag{38}$$

For $a = \sigma$ and ρ, we have

$$p_a = \frac{1}{2W}[(W^2 - M_{\pi\pi}^2 - m_N^2)^2 - 4M_{\pi\pi}^2 m_N^2]^{1/2}, \tag{39}$$

$$E_a(p_a) = [M_{\pi\pi}^2 + p_a^2]^{1/2}. \tag{40}$$

For the calculation of Eqs.(2) and (28), we thus present $< \rho N|T(W)|\pi N >$ and $< \sigma N|T(W)|\pi N >$ in the range $0 \leq p_a \leq p_{a,max}$ with

$$p_{a,max} = \frac{1}{2W}[(W^2 - (2m_\pi)^2 - m_N^2)^2 - 4(2m_\pi)^2 m_N^2]^{1/2}. \tag{41}$$

The above equations are for the calculations of the $\pi N \to \pi\pi N$ through the resonant $\pi\Delta$, σN and ρN channels. There are also weaker contributions from the direct production mechanisms, as illustrated in Fig. 1, which can be calculated by using the procedures explained in Ref. [**?**].

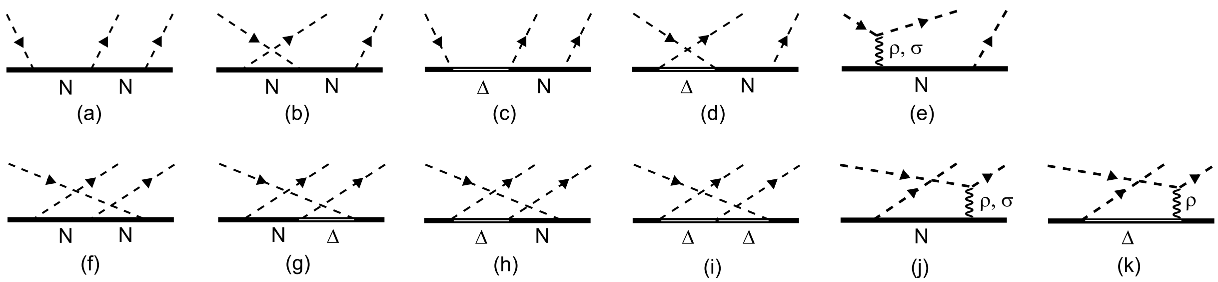

FIG. 1: The considered $v_{\pi N, \pi\pi N}$.