Formula for calculating the $N(e,e'\pi)N$ cross sections
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For the pion electroproduction, ($e(p_e)+N(p_N) \to e'(p'_e)+\pi(k)+N(p'_N)$), the differential cross section is conventionally written as

$$\frac{d\sigma}{dE'_e d\Omega_{e'} d\Omega_\pi} = \Gamma \frac{d\sigma^v}{d\Omega_\pi} \ , \tag{1}$$

where $Q^2 = -q^2$, $q = p_e - p'_e = (\omega_L, \boldsymbol{q}_L)$, $W = \sqrt{(p_N+q)^2}$, and

$$\Gamma = \frac{\alpha q_L^\gamma}{2\pi^2 Q^2} \frac{E'_e}{E_e} \frac{1}{1-\epsilon}. \tag{2}$$

Here, we have defined $\alpha = e^2/4\pi = 1/137$ and the effective photon energy in the laboratory system and ϵ are given as

$$q_L^\gamma = \frac{W^2 - m_N^2}{2m_N} \ , \tag{3}$$

$$\epsilon = [1 + \frac{2\boldsymbol{q}_L^2}{Q^2} \tan^2 \frac{\theta_e}{2}]^{-1}, \tag{4}$$

where θ_e is the angle between the outgoing and incoming electrons, and m_N is the nucleon mass and $\boldsymbol{q}_L$ is momentum transfer in the laboratory system.

The differential cross section $d\sigma^v/d\Omega_\pi$ in Eq. (1) is defined in final πN center of mass frame with the following coordinate system:

$$\hat{z} = \hat{q} = \frac{\boldsymbol{q}}{|\boldsymbol{q}|} \tag{5}$$

$$\hat{y} = \frac{\boldsymbol{q}\times\boldsymbol{k}}{|\boldsymbol{q}\times\boldsymbol{k}|} \tag{6}$$

$$\hat{x} = \hat{y}\times\hat{z} \tag{7}$$

We then have the following expression:

$$\frac{d\sigma^v}{d\Omega_\pi} = \frac{d\sigma_T}{d\Omega_\pi} + \epsilon\frac{d\sigma_L}{d\Omega_\pi} + \epsilon\frac{d\sigma_{TT}}{d\Omega_\pi}\cos 2\phi_\pi + \sqrt{2\epsilon(\epsilon+1)}\frac{d\sigma_{LT}}{d\Omega_\pi}\cos\phi_\pi$$
$$+ h_e\sqrt{2\epsilon(1-\epsilon)}\frac{d\sigma_{LT'}}{d\Omega_\pi}\sin\phi_\pi \ , \tag{8}$$

where h_e is the helicity of the incoming electron, ϕ_π is the pion angle measured from the $e-e'$ scattering plane of electron, and

$$\frac{d\sigma_T}{d\Omega_\pi} = \frac{|\boldsymbol{k}|}{|\boldsymbol{q}_\gamma|}\sum_{spin}\frac{F^{xx}+F^{yy}}{2} \ , \tag{9}$$

$$\frac{d\sigma_L}{d\Omega_\pi} = \frac{|\boldsymbol{k}|}{|\boldsymbol{q}_\gamma|}\sum_{spin}\frac{Q^2}{|\boldsymbol{q}|^2}F^{00} \ , \tag{10}$$

$$\frac{d\sigma_{TT}}{d\Omega_\pi} = \frac{|\boldsymbol{k}|}{|\boldsymbol{q}_\gamma|}\sum_{spin}\frac{F^{xx}-F^{yy}}{2} , \tag{11}$$

$$\frac{d\sigma_{LT}}{d\Omega_\pi} = \frac{|\boldsymbol{k}|}{|\boldsymbol{q}_\gamma|}\sum_{spin}(-1)\sqrt{\frac{Q^2}{|\boldsymbol{q}|^2}}Re(F^{x0}) , \tag{12}$$

$$\frac{d\sigma_{LT'}}{d\Omega_\pi} = \frac{|\boldsymbol{k}|}{|\boldsymbol{q}_\gamma|}\sum_{spin}\sqrt{\frac{Q^2}{|\boldsymbol{q}|^2}}Im(F^{x0}) . \tag{13}$$

Here $\boldsymbol{q}$ is the momentum transfer to the initial nucleon and $\boldsymbol{k}$ is the pion momentum in the center of mass system of the final πN state:

$$\omega = \frac{W^2 - M_N^2 - Q^2}{2W} , \tag{14}$$

$$|\boldsymbol{q}| = \sqrt{Q^2+\omega^2} , \tag{15}$$

$$|\boldsymbol{k}| = \sqrt{(\frac{W^2+m_\pi^2-M_N^2}{2W})^2 - m_\pi^2} , \tag{16}$$

and

$$|\boldsymbol{q}_\gamma| = \frac{W^2-M_N^2}{2W} . \tag{17}$$

Here ω and $\boldsymbol{q}_\gamma$ are the energy transfer and the effective photon energy in the center of mass system. Integrating pion angles, Eqs.(1) and (8) lead to

$$\frac{d\sigma}{dE'_e d\Omega_{e'}}(Q^2,W) = \Gamma[\sigma_T(Q^2,W) + \epsilon\sigma_L(Q^2,W)]. \tag{18}$$

where

$$\sigma_{T/L} = \sum_{\pi^+,\pi^0}\int d\Omega_\pi \frac{d\sigma_{T/L}}{d\Omega_\pi} \tag{19}$$

In the coordinate system defined by Eqs.(5)-(7), the pion momentum $\boldsymbol{k}$ is on the $x-z$ plane. We thus can defne

$$\hat{k} = \boldsymbol{k}/|\boldsymbol{k}| = \cos\theta\hat{z} + \sin\theta\hat{x} , \tag{20}$$

where θ is the angle between the outgoing pion and the virtual photon. The quantities F^{ij} with $i,j = x,y,0$ in Eqs. (9)-(13) are defined as

$$\sum_{spin} F^{ij} = \frac{1}{2}\sum_{m_{s_i},m_{s_f}} < m_{s_f}|\mathcal{F}^i|m_{s_i}>< m_{s_f}|\mathcal{F}^j|m_{s_i}>^* , \tag{21}$$

where m_s is the z-component of the nucleon spin, and $\mathcal{F}^i$ is defined by the Chew-Goldberger-Low-Nambu (CGLN) amplitude $\mathcal{F}_{CGLN} = \mathcal{F}^\mu\epsilon_\mu$.

The CGLN amplitude can be expressed in terms of Pauli operator $\boldsymbol{\sigma}$, $\hat{q}$, $\hat{k}$ and the photon polarization vector $\epsilon^\mu = (\epsilon_0, \boldsymbol{\epsilon})$

$$\begin{aligned}\mathcal{F}^\mu\epsilon_\mu = &-(i\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_\perp F_1 + \boldsymbol{\sigma}\cdot\hat{k}\boldsymbol{\sigma}\cdot\hat{q}\times\boldsymbol{\epsilon}_\perp F_2 + i\boldsymbol{\sigma}\cdot\hat{q}\hat{k}\cdot\boldsymbol{\epsilon}_\perp F_3 + i\boldsymbol{\sigma}\cdot\hat{k}\hat{k}\cdot\boldsymbol{\epsilon}_\perp F_4 \\ &+ i\boldsymbol{\sigma}\cdot\hat{q}\hat{q}\cdot\boldsymbol{\epsilon}F_5 + i\boldsymbol{\sigma}\cdot\hat{k}\hat{q}\cdot\boldsymbol{\epsilon}F_6) + i\boldsymbol{\sigma}\cdot\hat{k}\epsilon_0 F_7 + i\boldsymbol{\sigma}\cdot\hat{q}\epsilon_0 F_8 ,\end{aligned} \tag{22}$$

where $\boldsymbol{\epsilon}_\perp = \hat{q} \times (\boldsymbol{\epsilon} \times \hat{q})$. By using Eq.(20) and choosing $\epsilon^\mu = (\epsilon_0, \boldsymbol{\epsilon}) = (0,1,0,0), (0,0,1,0), (1,0,0,0)$ to evaluate Eq.(22), we then have

$$
\begin{aligned}
\mathcal{F}^x &= i\sigma_x(F_1 - \cos\theta F_2 + \sin^2\theta F_4 + i\sigma_z \sin\theta(F_2 + F_3 + \cos\theta F_4) \ , && (23)\\
\mathcal{F}^y &= i\sigma_y(F_1 - \cos\theta F_2) - \sin\theta F_2 \ , && (24)\\
\mathcal{F}^0 &= i\sigma_z(\cos\theta F_7 + F_8) + i\sigma_y \sin\theta F_7 \ . && (25)
\end{aligned}
$$

The amplitudes F_i are calculated from the multipole amplitudes $E_{l\pm}, M_{l\pm}, S_{l\pm}$ and $L_{l\pm}$ of the $\gamma^* + N \to \pi N$ process :

$$
\begin{aligned}
F_1 &= \sum_l [P'_{l+1}(x)E_{l+}(Q^2,W) + P'_{l-1}(x)E_{l-}(Q^2,W) + lP'_{l+1}(x)M_{l+}(Q^2,W) \\
&\quad + (l+1)P'_{l-1}(x)M_{l-}(Q^2,W)] \,, && (26)\\
F_2 &= \sum_l [(l+1)P'_l(x)M_{l+}(Q^2,W) + lP'_l(x)M_{l-}(Q^2,W)] \,, && (27)\\
F_3 &= \sum_l [P''_{l+1}(x)E_{l+}(Q^2,W) + P''_{l-1}(x)E_{l-}(Q^2,W) - P''_{l+1}(x)M_{l+}(Q^2,W) \\
&\quad + P''_{l-1}(x)M_{l-}(Q^2,W)] \,, && (28)\\
F_4 &= \sum_l [-P''_l(x)E_{l+}(Q^2,W) - P''_l(x)E_{l-}(Q^2,W) + P''_l(x)M_{l+}(Q^2,W) - P''_l(x)M_{l-}(Q^2,W)] \,, && (29)\\
F_5 &= \sum_l [(l+1)P'_{l+1}(x)L_{l+}(Q^2,W) - lP'_{l-1}(x)L_{l-}(Q^2,W)] \,, && (30)\\
F_6 &= \sum_l [-(l+1)P'_l(x)L_{l+}(Q^2,W) + lP'_l(x)L_{l-}(Q^2,W)] \,, && (31)\\
F_7 &= \sum_l [-(l+1)P'_l(x)S_{l+}(Q^2,W) + lP'_l(x)S_{l-}(Q^2,W)] \,, && (32)\\
F_8 &= \sum_l [(l+1)P'_{l+1}(x)S_{l+}(Q^2,W) - lP'_{l-1}(x)S_{l-}(Q^2,W)] \,, && (33)
\end{aligned}
$$

where $x = \hat{q} \cdot \hat{k}$, $P_L(x)$ is the Legendre function, $P'_L(x) = dP_L(x)/dx$ and $P''_L(x) = d^2P_L(x)/d^2x$. For the photo-production process $\gamma N \to \pi N$, the differential cross section is $d\sigma_T/d\Omega_\pi$ with $Q^2 = 0$.

The ANL-Osaka multipole amplitudes $E_{l\pm}(Q^2,W)$, $M_{l\pm}(Q^2,W)$, and $L_{l\pm}(Q^2,W)$ for $W = 1080 - 2000$ MeV and $Q^2 = 0 - 3$ (GeV/c)^2 are presented on the webpage.