

Cosmas Zachos

ANL/UC-JTI workshop

4/14/2009

[arXiv:0806.3515] PhysLettB666 (2008) 386-390, T Curtright, D Fairlie & CZ; [arXiv:0903.4889] PhysLettB, in press, T Curtright, X Jin, L Mezincescu, D Fairlie, & CZ

BASICS OF TERNARY ALGEBRAS
AND THEIR UNDERLYING NAMBU BRACKETS

Superposed M2-brane Lagrangean Model Chern-Simons interactions are predicated on Ternary algebras (CFZ, PhysLett B405 (1997) 37-44; Basu & Harvey, 2004; Bagger & Lambert, 2007)

Consider **associative** multiplication of 3 operators, fully antisymmetrized (Nambu 1973, Filippov 1984), the 3QNB,

$$[A, B, C] \equiv A [B, C] + B [C, A] + C [A, B] .$$

$= \frac{1}{2}\{A, [B, C]\} + \text{cyclic}$ \rightsquigarrow The trinomial knows about **anticommutators**.
Nontrivial trace.

It is related to, but, (**in sharp contrast** to 2N-QNBs vs. 2N-CNBs), it is **not a quantum deformation** of the also linear, antisymmetric 3CNB,

$$\{a, b, c\} = \epsilon^{ijk} \partial_i a \partial_j b \partial_k c = \frac{\partial(a, b, c)}{\partial(x, y, z)},$$

a Jacobian determinant (volume element).

► Will not consider the Awata-Li-Minic-Yoneya (1999) bracket,
 $\langle A, B, C \rangle \equiv [A, B] \text{Tr}C + [B, C] \text{Tr}A + [C, A] \text{Tr}B,$
repackaged commutators — traceless.

REVIEW OF A_4

⊙ Nambu noted that the $su(2)$ Casimir appears in the 3QNB,

$$[L_x, L_y, L_z] \equiv L_x [L_y, L_z] + L_y [L_z, L_x] + L_z [L_x, L_y] = i (L_x^2 + L_y^2 + L_z^2) = iL^2 .$$

\rightsquigarrow the BL A_4 is in the enveloping algebra for $SU(2)$:

$$Q_x = \frac{L_x}{\sqrt[4]{L^2}}, \quad Q_y = \frac{L_y}{\sqrt[4]{L^2}}, \quad Q_z = \frac{L_z}{\sqrt[4]{L^2}}; \quad Q_t \equiv \sqrt[4]{L^2},$$

yields

$$[Q_x, Q_y, Q_z] = iQ_t, \quad [Q_t, Q_x, Q_y] = iQ_z, \quad [Q_t, Q_y, Q_z] = iQ_x, \quad [Q_t, Q_z, Q_x] = iQ_y.$$

Summarized as

$$[Q_a, Q_b, Q_c] = i\epsilon_{abc}{}^d Q_d ,$$

where $\epsilon_{xyzt} = +1$ with a $[-1, -1, -1, +1]$ Lorentz signature.

↷ Amusing Aside: A_3

There is an even smaller 3QNB subalgebra, of this, namely

$$[Q_x, Q_y, Q_z \pm Q_t] = \pm i(Q_z \pm Q_t) ,$$

↷ e.g.,

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix} \right] = \sqrt{3} \begin{pmatrix} 1 & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix},$$

etc...

THE FI CONDITION

Filippov's (1984) special condition, "FI",

$$[D, E, [A, B, C]] = [[D, E, A], B, C] + [A, [D, E, B], C] + [A, B, [D, E, C]]$$

is not a general identity for associative operators, much unlike the Jacobi identity for commutators (2QNBs), and, unlike them, is **not** antisymmetric in all indices.

► But it **is an identity for 3CNBs**, and follows from the Leibniz rule of the **derivations** involved, and combinatorics: $\epsilon^{ab}[c_{\epsilon}^{def}] = 0$
(TC & CZ, NewJPhys 4 (2002) 83.1-83.16).

It happens to be satisfied for **very few** 3QNB-based ternary algebras: for finite sets of operators, only A_4 and A_3 ; and for a few infinite ones (below). Is needed for BL model-building (consistency of supersymmetry). Defines Filippov n-Lie Algebras.

Informal preview:

⊗ **All such known ones are basically isomorphic to 3CNBs**

○ If you fully antisymmetrize 3QNB-into-3QNB, you get 5QNB, usually undefined (TC & CZ, PhysRev D68 (2003) 085001); and 3CNB-into-3CNB yields 5CNB.

↯ But if the FI is not "Jacobi", where is "Jacobi"?

BREMNER IDENTITIES

Bremner (1998, 2006), Nuyts (2008, unpublished) confirmed that there are no degree-5 (3QNB-into-3QNB) identities; but found there is a degree-7 (3QNB-into-3QNB-into-3QNB) identity,

$$[[A, [B, C, D], E], F, G] = [[A, B, C], [D, E, F], G] \quad \circlearrowright ,$$

\circlearrowright : for fixed A , and antisymmetrized B, C, D, E, F, G .

A ternary algebra one might define through 3QNBs for associative chains of operators,

$$[T_a, T_b, T_c] = if_{abc}{}^d T_d ,$$

does not exist unless it satisfies this identity. (e.g. \Leftrightarrow constraining structure constants and centers).

\rightsquigarrow If it does, it **need not satisfy the FI condition**—which is often harder to achieve.

AN EASY WAY TO SATISFY FI

In general checking FIs for a system is cumbersome—and the answer is usually negative...

► However, **if** a ternary algebra of 3QNBs shares structure constants with a ternary algebra of 3CNBs, **by identical combinatorics**, it will **also** satisfy the FI, because that is an identity for 3CNBs; which, in turn, likewise satisfy the Bremner identities, since the 3QNB version does.

⊛ One need not actually find an explicit associative operator realization of the 3CNB (analogous to the commutator realization of Poisson Brackets, $f(q,p) \mapsto \nabla f \times \nabla$ —very hard indeed!). One **only** need find an abstract **shadow isomorph**: **a 3CNB with the same structure constants**.

E.g., for finite-dimensional algebras, such as A_4 ,

$$Q_{x,y,z,t} \quad \mapsto \quad \sqrt{zx}, \quad \sqrt{zy}, \quad z, \quad \frac{x^2 + y^2}{2} .$$

For A_3 ,

$$\sqrt{zx}, \quad \sqrt{zy}, \quad \sqrt{zz}.$$

Satisfy BI and FI. Comparably interesting ones in this class are some infinite-dimensional ones:

THE FOLLOWING SATISFY BI AND FI TOO

For the most general 3CNB on a T^3 basis, $e_a \equiv \exp(a \cdot (x, y, z)), \dots$, up to normalization, the tetrahedron-volume algebra,

$$[E_a, E_b, E_c] = a \cdot (b \times c) E_{a+b+c} .$$

In fact, in the $\{e_a, e_b, e_c\}$ realization, it is **easier to check both** the FI and the BI!

Now taking a subalgebra, from 3-fold infinity of indices to a 2-fold one, e.g. for the **closing** set of functions,

$$w_m^a(x, y, z) \equiv \sqrt{z} \exp((a + 1/2)x + my),$$

yields the T^2 algebra of Chakraborty, Kumar, & Jain, also satisfying FI and BI.

► Finally, one understands the reason that the Virasoro-Witt ternary algebra with a 1-fold infinity, S^1 , of indices, a smaller subalgebra of the tetrahedral volume one, satisfies the FI (BI evident).

This ternary algebra is put together by Witt (centerless Virasoro) and Goddard-Thorn operators,

$$[Q_k, Q_m, Q_n] = (k - m) (m - n) (k - n) R_{k+m+n} ,$$

$$[Q_p, Q_q, R_k] = (p - q) \left(Q_{k+p+q} + s k R_{k+p+q} \right) ,$$

$$[Q_p, R_q, R_k] = (k - q) R_{k+p+q} , \quad [R_p, R_q, R_k] = 0 ,$$

with s a parameter. Only for $s = \pm 2i$ is the FI satisfied.

We can understand this otherwise baffling fact as follows.

Simplify the algebra by $L_m \equiv Q_m + m \frac{s - \sqrt{s^2 + 4}}{2} R_m$, \rightsquigarrow

$$[L_p, L_q, L_k] = 0 , \quad [R_p, R_q, R_k] = 0 , \quad [L_p, R_q, R_k] = (k - q) R_{k+p+q} ,$$

$$[R_p, L_q, L_k] = - (k - q) \left(L_{k+p+q} + p \sqrt{s^2 + 4} R_{k+p+q} \right) .$$

In fact, the square root may be absorbed in the relative normalizations of the R s and L s, and could be set to **one**; unless it were **zero**, which is thus a Wigner-Inonü contraction of that general case. This is the interesting case, satisfying the FI, and possessing a L–R $O(2)$ symmetry,

$$[L_p, L_q, L_k] = 0, \quad [R_p, R_q, R_k] = 0,$$

$$[L_p, R_q, R_k] = (k - q) R_{k+p+q}, \quad [R_p, L_q, L_k] = -(k - q) L_{k+p+q}.$$

In this case, the closing set

$$L_m = x e^{mz}, \quad R_m = y e^{mz},$$

identifies the underlying 3CNB algebra and so explains the FI.

▲ Are **all** FI-compliant ternary algebras really 3CNBs in disguise?

