Dyson's instability in lattice models

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done in part with A. Denbleyker, D. Du, Y. Liu and A. Velytsky

Content of the talk

- Main questions addressed
- Dyson instability, complex coupling and large fields
- Lattice perturbation theory and density of states (arXiv:0807.0185 [hep-lat], Phys. Rev. D78 054503)
- Sigma models in the large-N limit, in the complex coupling plane (preprint in progress)
- Conclusions

Main Questions Addressed

What is the large order behavior of perturbative series in lattice gauge theory?

How do truncated series compare with numerical data?

Can we treat these questions by calculating the density of states (color entropy)?

Can we solve or understand better these problems in the large-N limit?

In asymptotically free theories with no phase transitions at real coupling, can weak coupling expansions be used to approach the large distance behavior?

Few facts about pure gauge lattice theory:

- Existing series (average plaquette up to order 16 in g^2) indicate power growth (not factorial!)
- Partition function well defined when $g^2 < 0$
- Mass gap, no phase transition, between weak and strong coupling for $SU(2)$ and $SU(3)$ in four dimensions
- The problem is fully understood for the one plaquette model

Dyson's instability

Suppose that a physical quantity in QED can be calculated as a perturbative series $F(e^2) = a_0 + a_1 e^2 + \dots$.

If we assume that the series has a finite radius of convergence, then, for e^2 sufficiently small, we can interpret $F(-|e^2|)$ as the value of this quantity in a fictitious world where same charge particles attract. But in this fictitious world, every physical state is unstable. So, the radius of convergence is zero.

"The argument [...] is lacking in mathematical rigor and in physical precision. It is intended to be suggestive, to serve as a basis for further discussions" (F. J. Dyson, Phys. Rev. 85, 631 (1952))

Complex coupling (Bender, Wu, Zinn-Justin, Parisi, Brezin...)

The validity of Dyson conclusions were confirmed for the anharmonic oscillator, the double-well potential and other models.

Dispersion relations $+$ semi-classical calculations at small negative λ or e^2 predict the large order behavior of (asymptotic) series.

Theories with stable states at negative coupling can be constructed (Carl Bender et al.)

Large-N: some quantities (e. g. , ground state energy, anomalous dimensions ...) have a finite radius of convergence in the 't Hooft coupling (in the planar approximation). However Dyson instability is invoked by Polyakov (arXiv 0709.2899) in the AdS/CFT context.

Asymptotic series and large fields (YM, PRL88:141601)

$$\int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} \neq \sum_0^\infty \frac{(-\lambda)^l}{l!}\int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2}\phi^{4l}$$

The peak of the integrand of the r.h.s. moves too fast when the order increases. On the other hand, if we introduce a field cutoff, the peak moves outside of the integration range and

$$\int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} = \sum_0^\infty \frac{(-\lambda)^l}{l!}\int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2}\phi^{4l} \qquad (1)$$

General expectations: for a finite lattice, the partition function Z calculated with a field cutoff is convergent and $\ln(Z)$ has a finite radius of convergence. ϕ_{max} is an optimization parameter fixed using strong coupling, for instance.

Quenched lattice QCD

Lattice gauge theories with a compact group (e.g., Wilson's lattice QCD) have a build-in large field cutoff: the group elements associated with the links are integrated with dU_l the compact Haar measure. N is the number of colors. UV and large field regularization preserve gauge invariance.

$S = \sum_{plaq.}(1 - (1/N)ReTr(U_p))$

$\beta = 2N/g^2$

$Z = \prod_l \int dU_l \mathrm{e}^{-\beta S}$

Number of plaquettes: $\mathcal{N}_p \equiv \ L^D D(D-1)/2$

Average plaquette: $P(\beta) = (1/\mathcal{N}_p) \left\langle \sum_p (1 - (1/N)ReTr(U_p)) \right\rangle$

The density of states

$Z(\beta)$ is the Laplace transform of $n(S)$, the density of states

$$Z(\beta) = \int_0^{S_{max}} dS\ n(S)\ \mathrm{e}^{-\beta S}\ ,$$

with

$$n(S) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p)))$$

ln$(n(S))$ is a "color entropy" ($\propto\ \mathcal{N}_p$, extensive); $n(S) = \mathrm{e}^{\mathcal{N}_p f(S/\mathcal{N}_p)}$

$S_{max} = 2\mathcal{N}_p$ for $SU(2N)$, $\frac{3}{2}\mathcal{N}_p$ for $SU(3)$; ($\mathcal{N}_p$: number of plaquettes)

One plaquette ($SU(2)$)

$Z(\beta) = \int_0^2 dS n(S) \mathrm{e}^{-\beta S} = 2\mathrm{e}^{-\beta} I_1(\beta)/\beta$ (analytical in the entire β plane)

$n(S) = \frac{2}{\pi}\sqrt{S(2-S)}$ (invariant under $S \rightarrow 2 - S$)

The large order of the weak coupling expansion $\beta \rightarrow \infty$ is determined by the behavior of $n(S)$ near $S = 2$, itself probed when $\beta \rightarrow -\infty$ in agreement with the common wisdom that the large order behavior of weak coupling series can be understood in terms of the behavior at small negative coupling.

$\sqrt{2-S}$ is easy to approximate near $S = 0$ (radius of convergence $= 2$)

$Z(\beta) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{l=0}^{\infty} (2\beta)^{-l} \frac{\Gamma(l+1/2)}{l!(1/2-l)} \int_0^{2\beta} dt \mathrm{e}^{-t} t^{l+1/2}$ is convergent

The crucial step

$\int_0^{2\beta} dt \mathrm{e}^{-t} t^{l+1/2} \simeq \int_0^{\infty} dt \mathrm{e}^{-t} t^{l+1/2} + O(\mathrm{e}^{-2\beta})$ is responsible for the factorial behavior

The peak of the integrand crosses the boundary near order 2β

Dropping higher order terms (than order $\simeq 2\beta$) agrees with the rule of thumb (minimizing the first contribution dropped)

The non-perturbative part can be fully reconstructed (higher orders + "tails", PRD 74 096005)

L^4 lattices

The crossing is near order $2\beta\mathcal{N}_p$ which explains that up to order 16, no sign of factorial growth is seen on 8^4 and 24^4 lattices. However the tail effects may be important for reduced models (small volume models introduced in the context of the large-N limit).

Complex singularities for $|S| < S_{max}$ should explain the behavior of perturbative series at large volume.

Non-perturbative effects should be explainable by the contributions near S_{max} which can be probed at small negative coupling.

Z remains an analytic function of β in the entire complex β plane and the strong coupling expansions is dominated by the zeros of the partition function.

Lattice Perturbation Theory ($SU(3)$)

$P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \ldots .$

(F. Di Renzo et al. JHEP 10 038, P. Rakow Lat. 05)

Series analysis suggests a singularity: $P \propto (1/5.74 - 1/\beta)^{1.08}$ (Horsley et al, Rakow, Li and YM)

This means that the coefficients we know grow like 5.74^n rather than n!

Not seen in 2d derivative of P (would requires massless glueballs!)

Solution: complex singularities slightly off the real axis (PRD 73 036006)

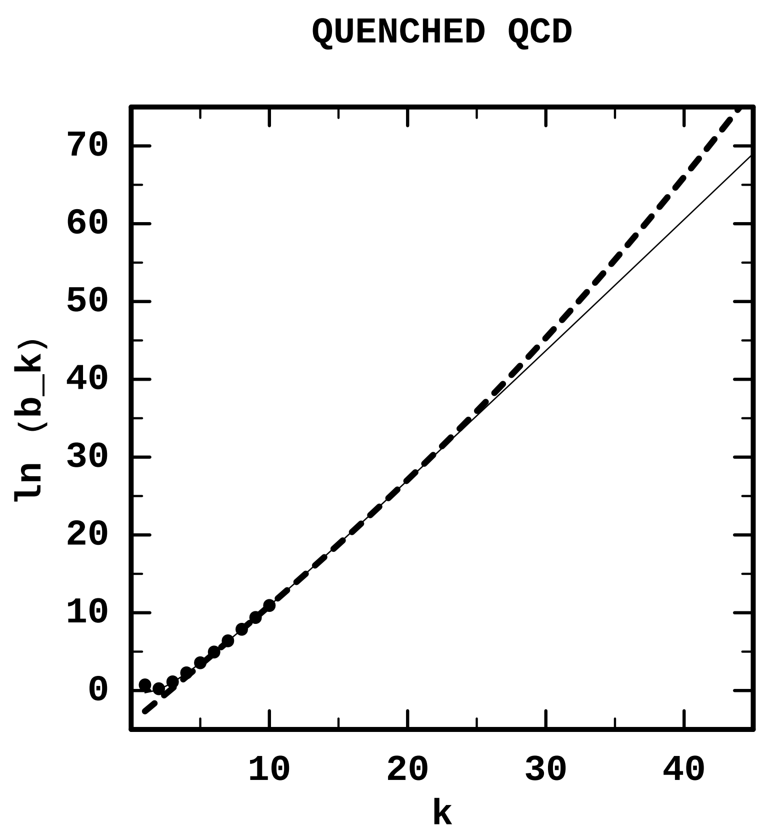


Figure 1: $\ln(b_k)$ for the mean field model (solid line) and the renormalon model (dashes). The dots up to order 10 are the known values. The two models yields similar coefficients up to order 20. After that, the integral model (renormalons) has the logarithm of its coefficients growing faster than linear.

Fisher's zeros

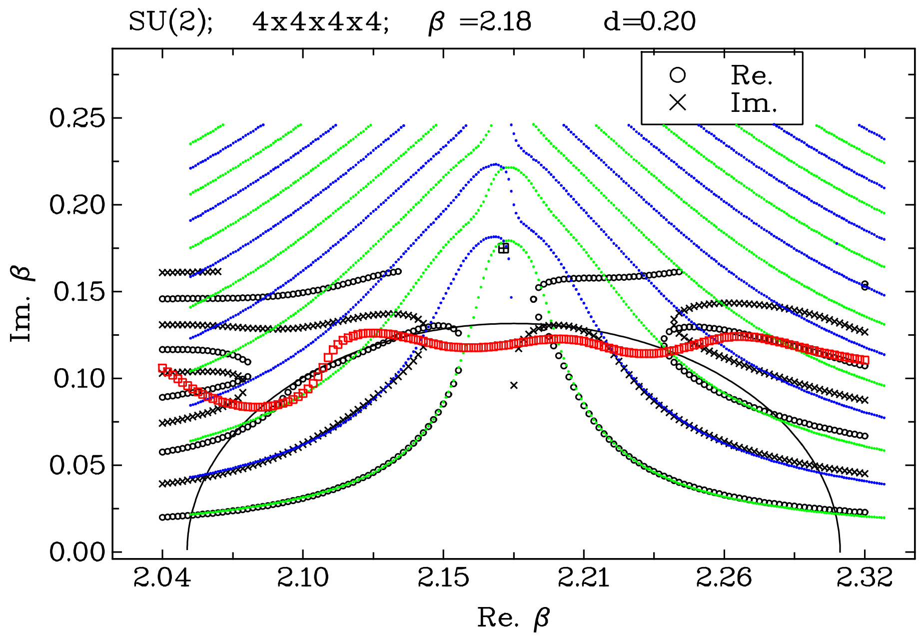


Figure 2: Zeros of the real (crosses) and imaginary (circles) using MC on a 4^4 lattice, for $SU(2)$ at $\beta_0 = 2.18$. The values for the real (green) and imaginary (blue) parts are obtained from a 4 parameter model.

A $SU(2)$ duality ($g^2 \rightarrow -g^2$ means $S \rightarrow 2\mathcal{N}_p - S$)

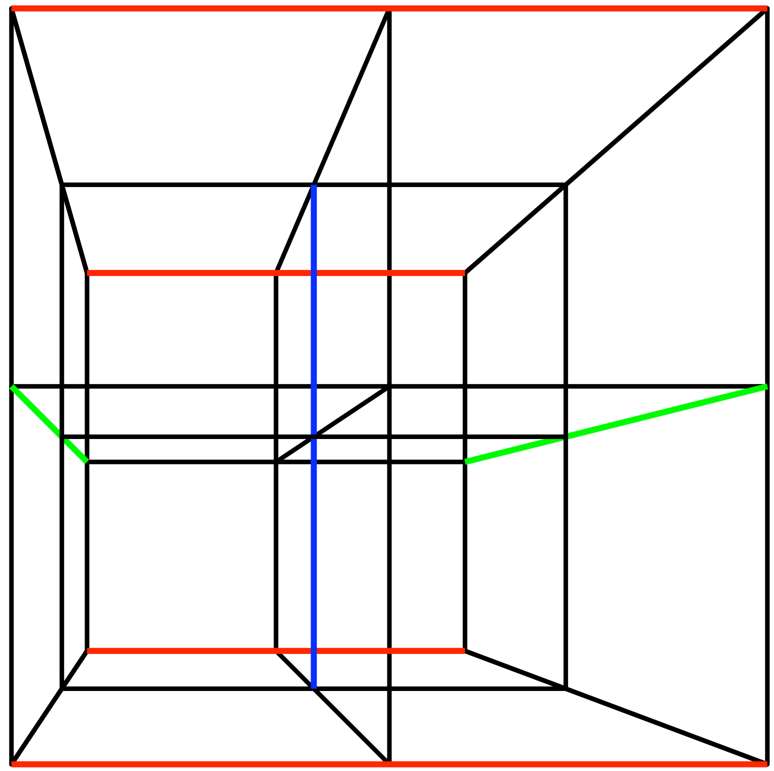
For cubic lattices with even number of sites in each direction and a gauge group that contains -1, it is possible to change $\beta ReTrU_p$ into $-\beta ReTrU_p$ by a change of variables $U_l \rightarrow -U_l$ on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette (Li, YM PRD71 016008). This implies

$$Z(-\beta) = \mathrm{e}^{2\beta\mathcal{N}_p} Z(\beta)$$

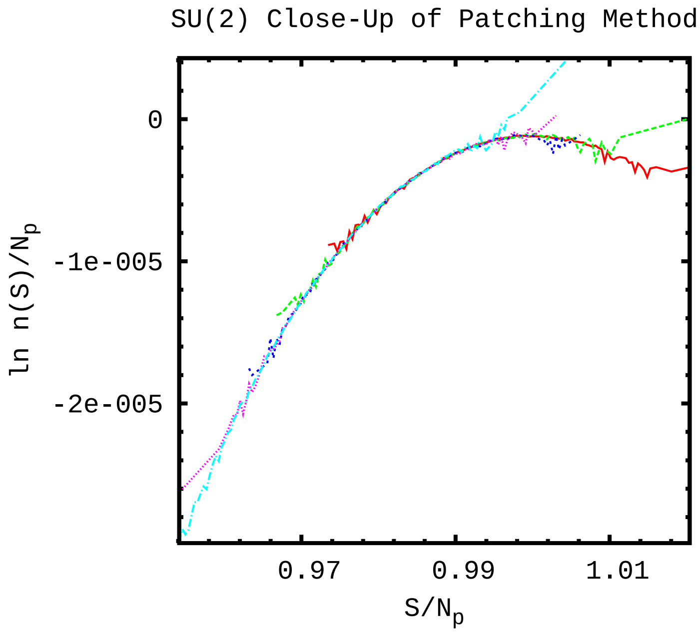
$$n(2\mathcal{N}_p - S) = n(S)$$

Thanks to this symmetry, we only need to know $n(S)$ for $0 \leq S \leq \mathcal{N}_p$ ($< S >= \mathcal{N}_p$ means $< TrU_p >$=0). Note: this is not a symmetry of the 1^4 reduced EK model.

For $D = 3$, an example of $\mathcal{L}$ is $\{(A,0,0),(0,A,1),(1,1,A)\}$ with A arbitrary. It is not difficult to show that there are 8 distinct $\mathcal{L}$.



Numerical calculation of $n(S)$ (A. denBleyker)



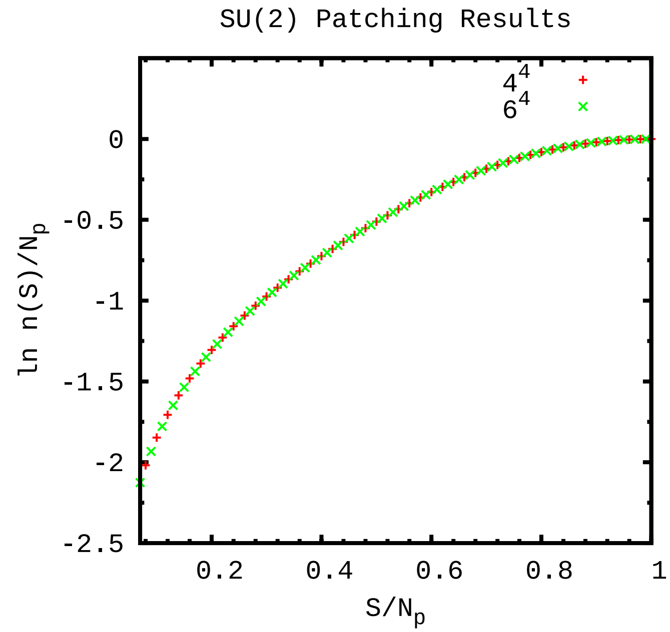


Figure 3: Results of patching $P_\beta(S)\mathrm{e}^{\beta S}$ for 4^4 and 6^4 (Phys. Rev. D78 054503); Volume effects will be discussed tomorrow.

Weak and strong coupling expansions

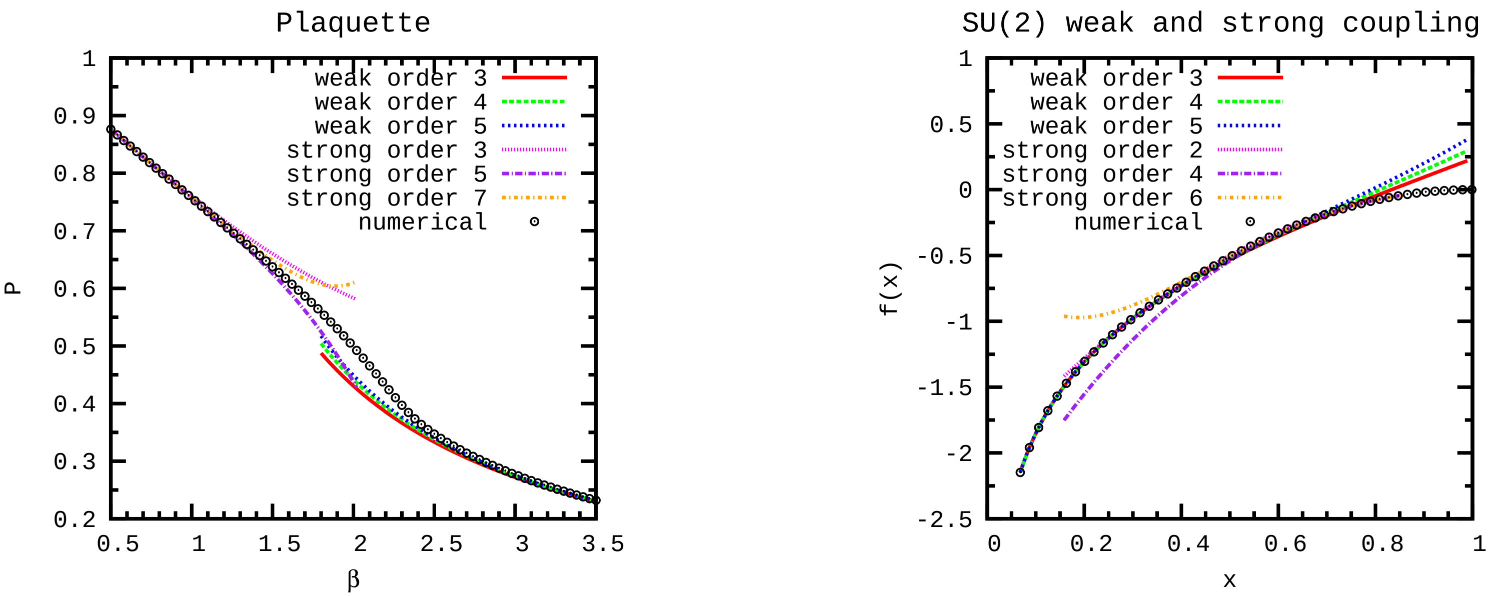


Figure 4: Average plaquette (left) and $ln(n(S))/\mathcal{N}_p$ (right) compared to weak and strong coupling expansions $(x = S/\mathcal{N}_p)$.

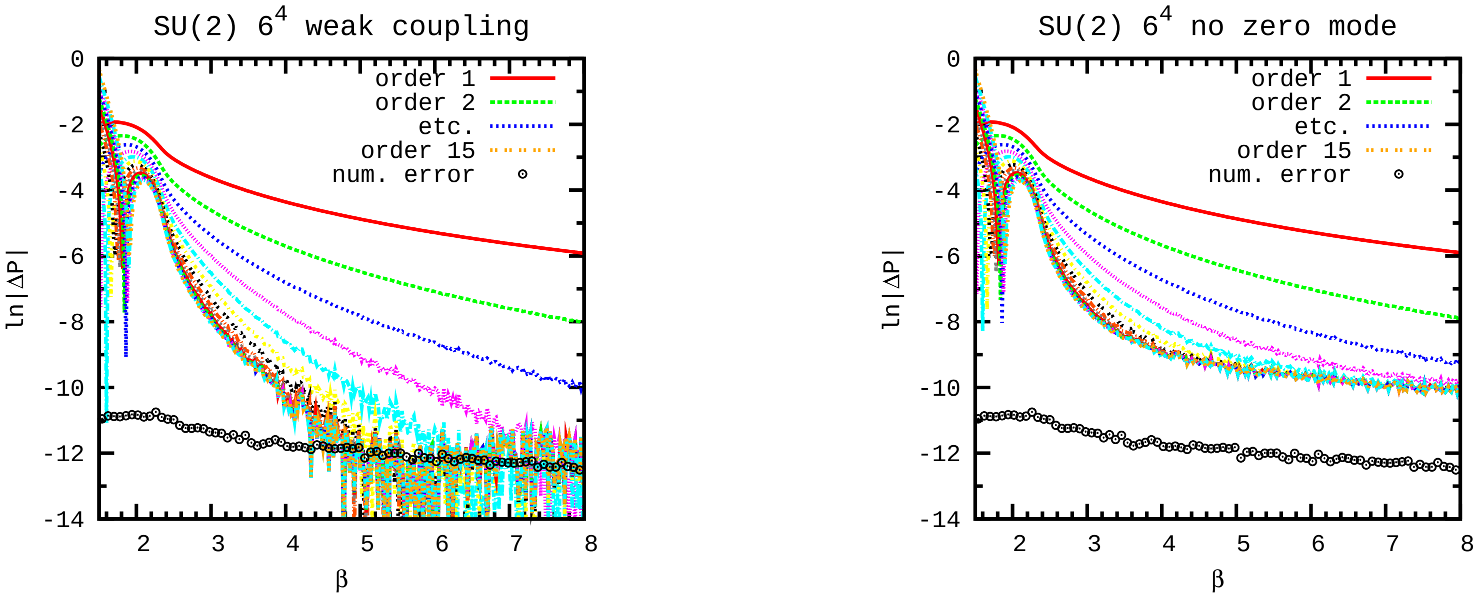


Figure 5: Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of P at successive orders (left) and without the zero mode (right).

Finite radius of convergence (strong coupling)

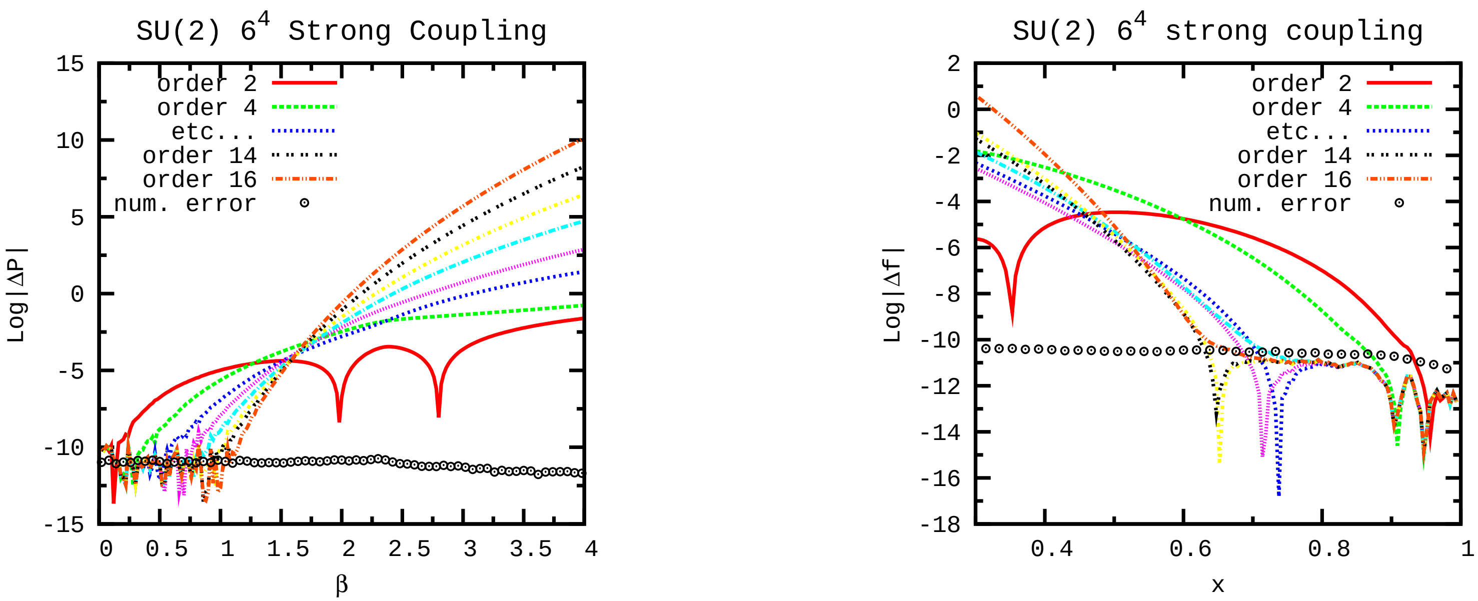
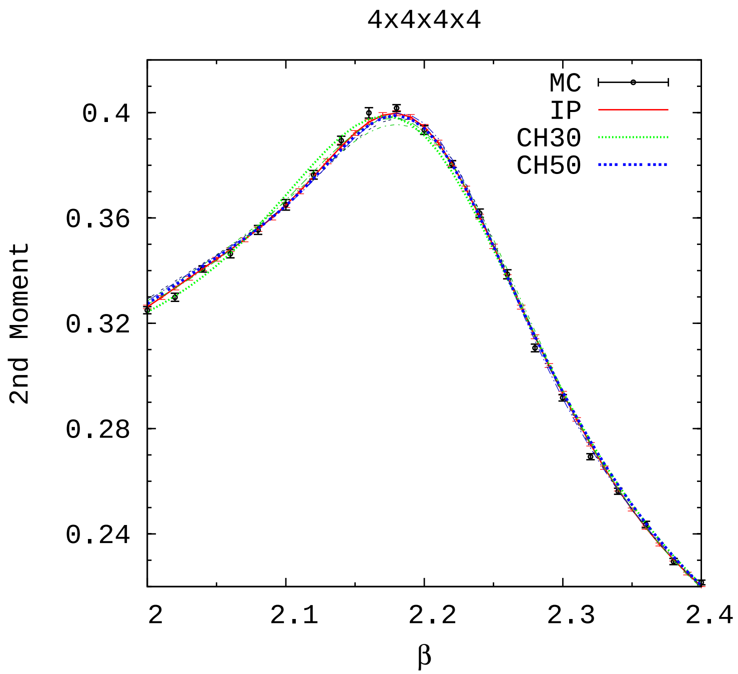


Figure 6: Logarithm of the absolute value of the difference between the numerical data and the strong coupling expansion of P (left) and f (right) at successive orders. For reference, we also show the numerical errors.

Moments (D. Du)



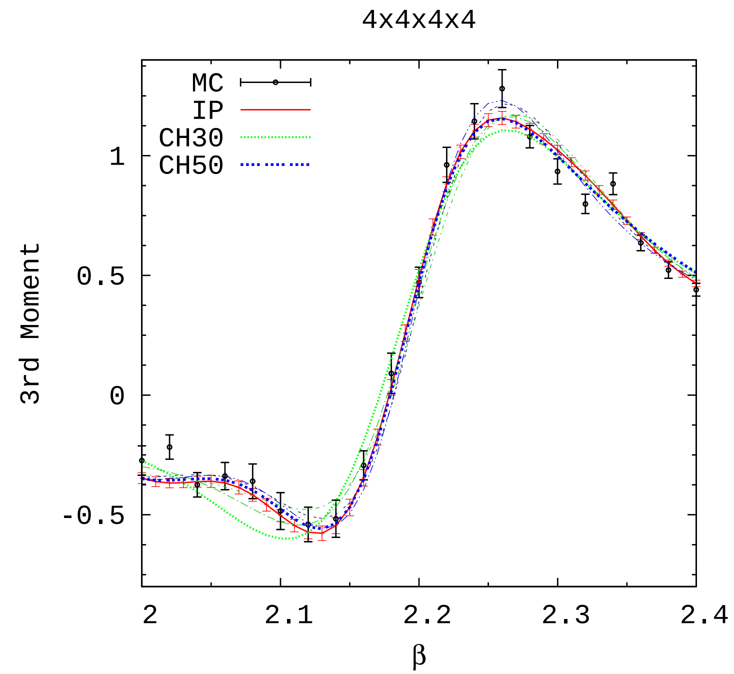


Figure 7: Comparison of the second and third moment calculated from the density of states and the direct MC result .

$U(1)$ lattice gauge theory

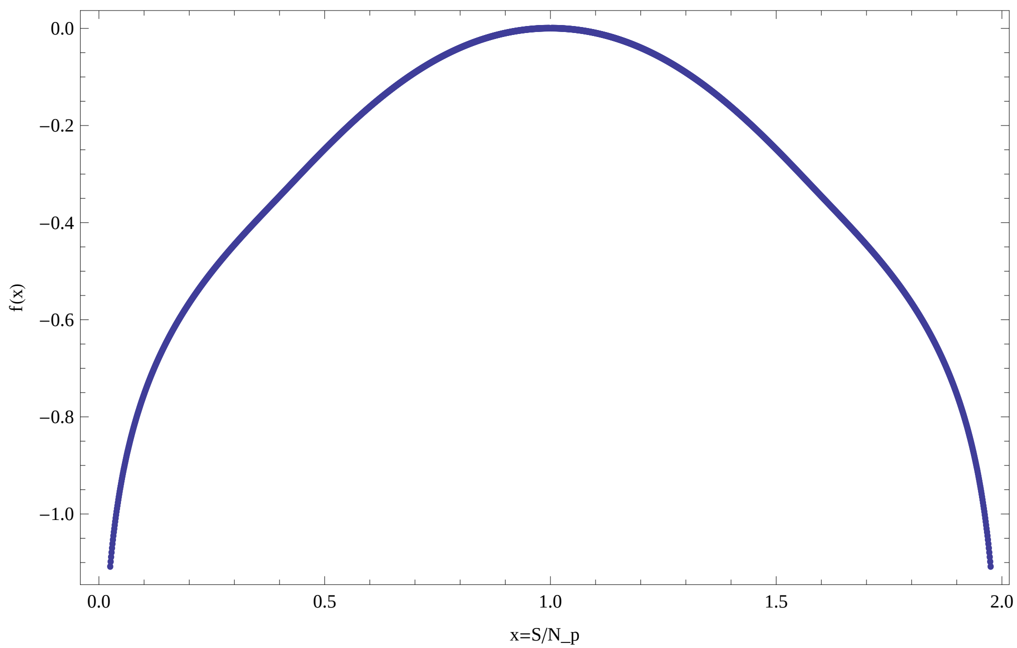


Figure 8: Density of states for $U(1)$ on a 4^4 lattice by multicanonical methods (A. Bazavov).

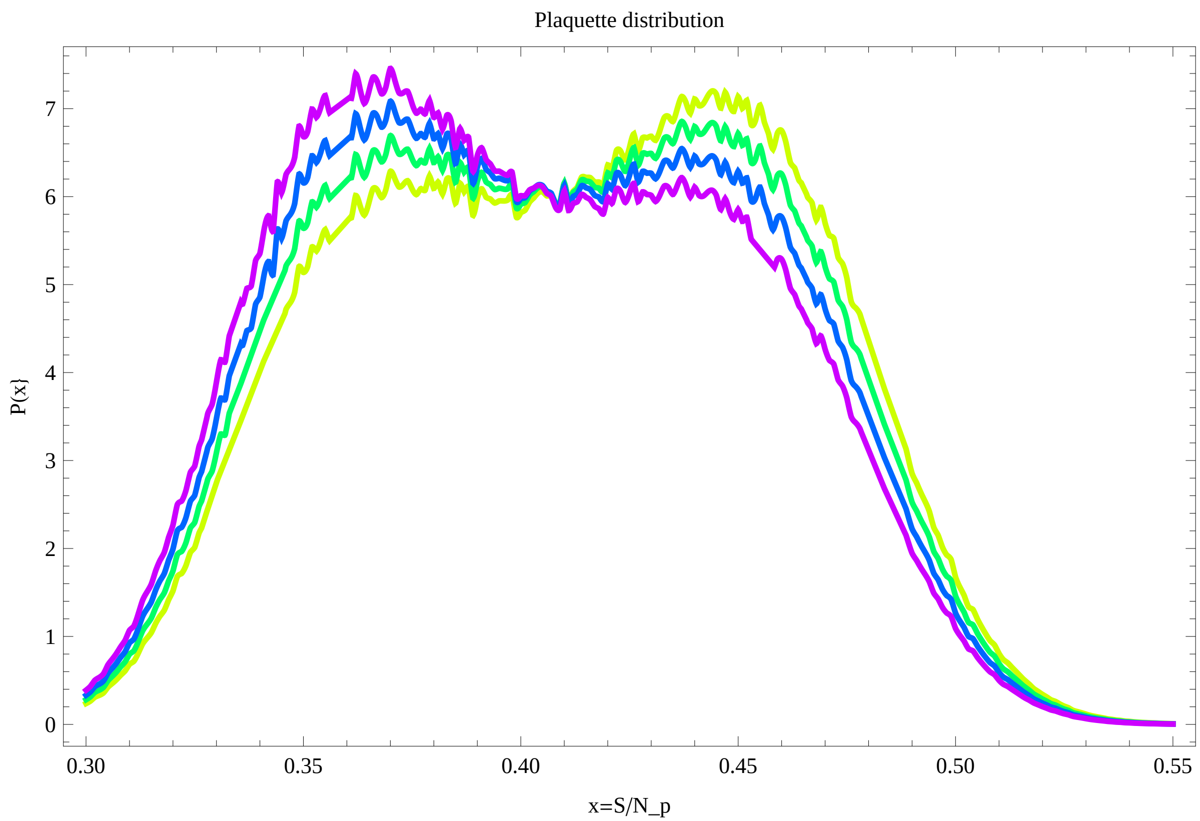


Figure 9: Plaquette distribution for $U(1)$ at β=0.978 (olive) , 0.979 (green), 0.98 (blue), and 0.981 (purple), using the density of states for a 4^4 lattice.

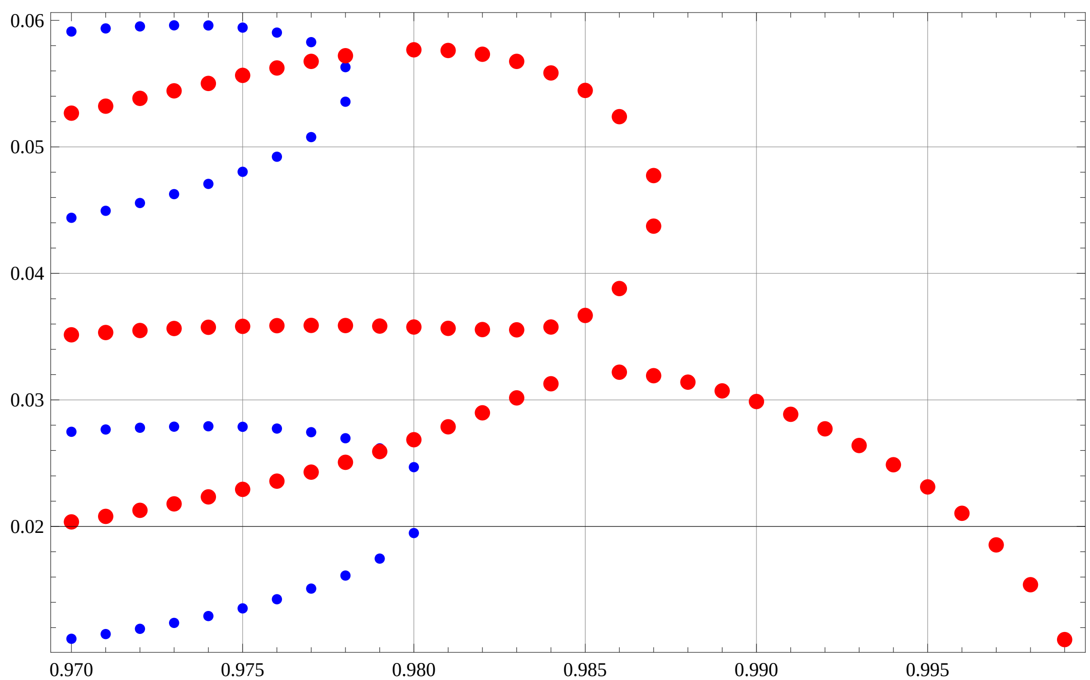


Figure 10: Zeros of Re and Im part of Z for $U(1)$ using the density of states for a 4^4 lattice. Real part of leading zero is about 0.979. As the volume increases, the zero gets closer (tomorrow's talk)

Linear $O(N)$ σ-model with a sharp momentum cutoff

$Z = \int \mathcal{D}\phi \mathrm{e}^{-\int d^D x[(1/2)(\partial\phi)^2 + (1/2)m_B^2\phi^2 + (\lambda^t/N)(\phi^2)^2 + (\eta^t/N^2)(\phi^2)^3]}$

After introducing auxiliary fields and integrating over the N vector $\vec{\phi}$

$Z \propto \int_{c-i\infty}^{c+i\infty} dM^2 \int_0^\infty dX \mathrm{e}^{-NV\mathcal{A}}$

$\mathcal{A} = (1/2)\int_{|k|\leq 1} \ln(k^2 + M^2) - (1/2)M^2 X + U(X)$

with $U(X) = (1/2)m_B^2 X + \lambda^t X^2 + \eta^t X^3 + \ldots$

Every dimensional quantity is expressed in units of a sharp cutoff

(ref: David et al. PRL 53, 2071)

Saddle point (gap equation)

$$\int_{|k|\leq 1} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2+M^2} = X \text{ and } M^2 = 2U'(X)$$

For a quartic potential

$$\int_{|k|\leq 1} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2+M^2} = (M^2 - m_B^2)/(4\lambda^t)$$

For D=2, when λ^t is positive, this equation has 1 solution. When λ^t is negative, this equation may have 2, 1 or no real solution. We call λ_c^t the value where the two real solutions coalesce and disappear if the coupling becomes more negative. (See figure)

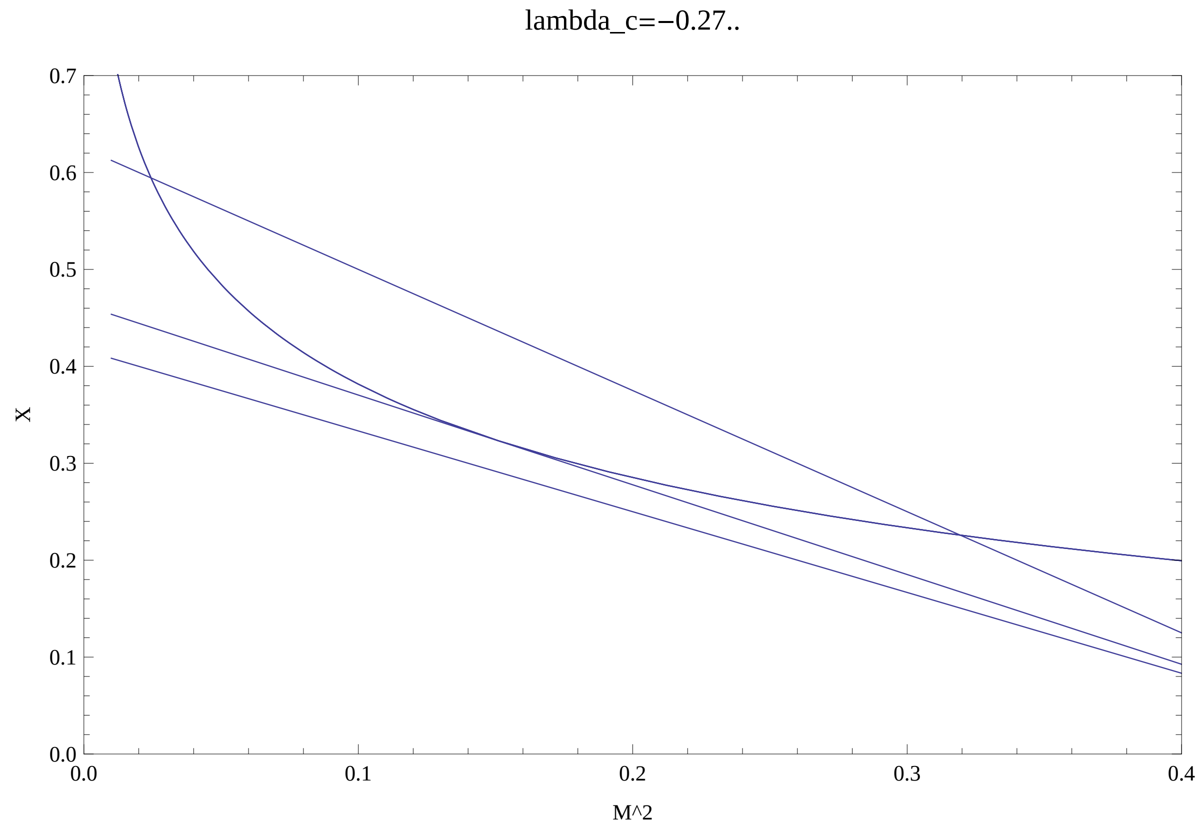


Figure 11: Graphical representation of the saddle point equation for $D = 2$ and m_B^2=0.5, with λ^t above, below and very close to λ_c^t

Perturbative solution

Perturbative solution: $M^2(\lambda^t) = m_b^2 + c_1\lambda^t + \dots$,

For $m_B^2 = 0.5$, $c_n/c_{n+1} \simeq -0.27(1 + 1.6/n)$ indicates a finite radius of convergence with a square root singularity near λ_c^t.

Numerical calculations with $m_B^2 = 0.5$ yield $\lambda_c^t \simeq -0.27$.

At large M^2, $M^2 \propto \sqrt{\lambda^t}$, in contrast to the nonlinear sigma model where $M^2 \propto \lambda^t$.

A study of the quadratic fluctuations shows that $\int ln(k^2 + M^2)$, induces a local minimum for the "master field" X when λ^t is not too negative. This is a classical result that is blind to the quantum tunneling (metastability, large field behavior).

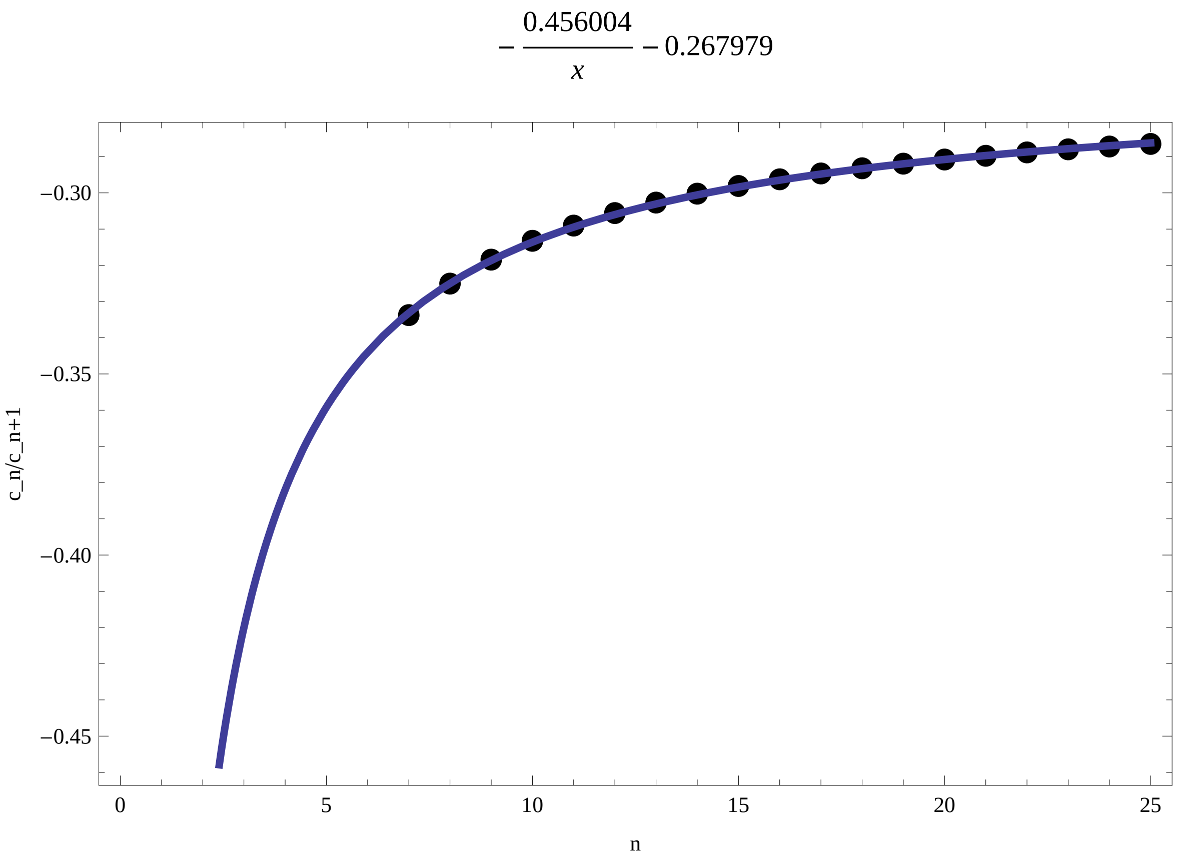


Figure 12: Ratios c_n/c_{n+1} and fit.

Nonlinear $O(N)$ sigma model on a square lattice

$Z = \int \prod_x d^N\phi_x \delta(\vec{\phi}_x \vec{\phi}_x - 1)\ \mathrm{e}^{-(1/g_0^2)E[\{\phi\}]}$

with $E[\{\phi\}] = \sum_{x,e}(1 - \vec{\phi}_x \vec{\phi}_{x+e})$

We assume a cubic lattice with an even number of sites in each directions and periodic boundary conditions. Under these conditions (as for $SU(2N)$ LGT)

$Z[-g_0^2] = \mathrm{e}^{4DL^D/g_0^2} Z[g_0^2]$

This can be seen by changing variable $\phi \rightarrow -\phi$ on sublattices with lattice spacing twice larger and such that they share exactly one site with each link of the original lattice.

Gap equation

$$\prod_{j=1}^{D} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \frac{1}{2(\sum_{j=1}^{D}(1-\cos(k_j))+M^2} = 1/\lambda^t \equiv b$$

with $\lambda^t = g_0^2 N$ kept constant as N becomes large.

The saddle point equation is invariant under $\lambda^t \rightarrow -\lambda^t$ together with $M^2 \rightarrow -M^2 - 4D$. This can be seen by changing variables $k_j \rightarrow k_j + \pi$ for all j.

For $D = 2$, $\lambda^t \rightarrow 0$ when $M^2 \rightarrow \ 0,\ -8\ , -4 \pm i\epsilon$ with double poles at $(k_1, k_2) = (0,0),\ (\pi,\pi),\ (0,\pi),\ (0,\pi)$ respectively.

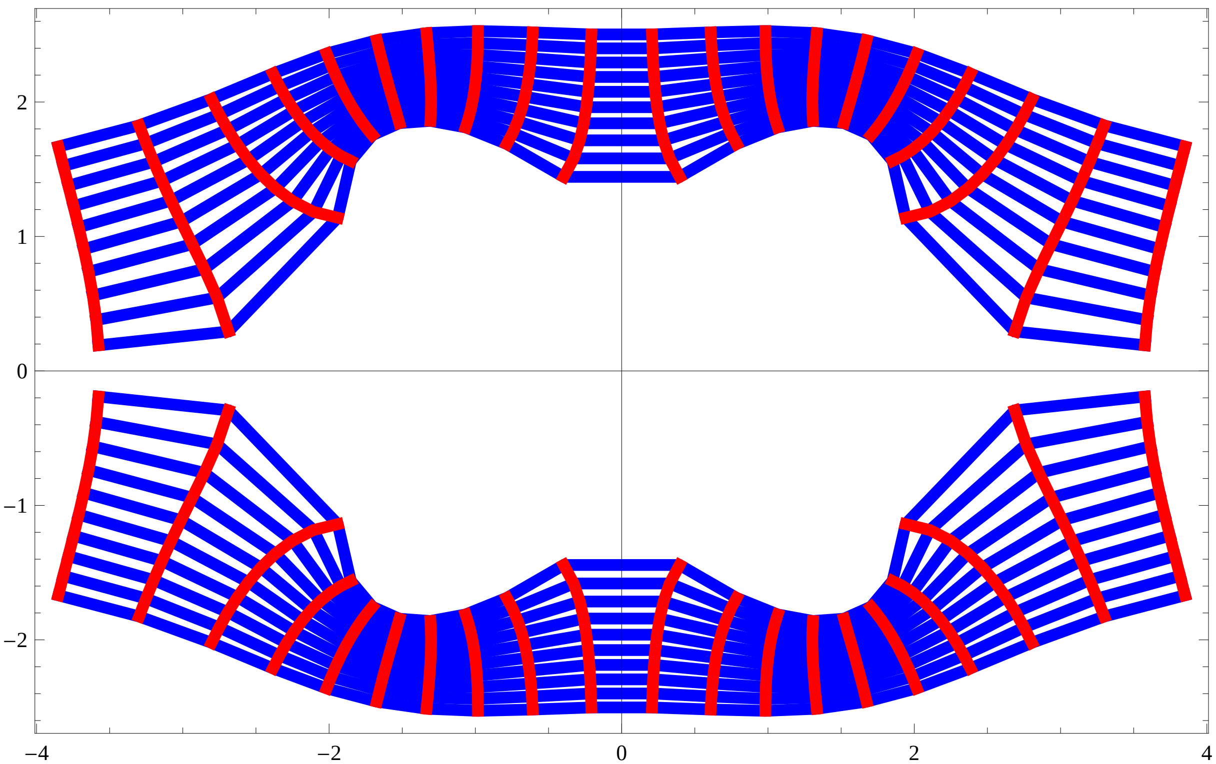


Figure 13: Complex values taken by λ^t when M^2 varies over the complex plane (here on horizontal and vertical lines in the M^2 plane).

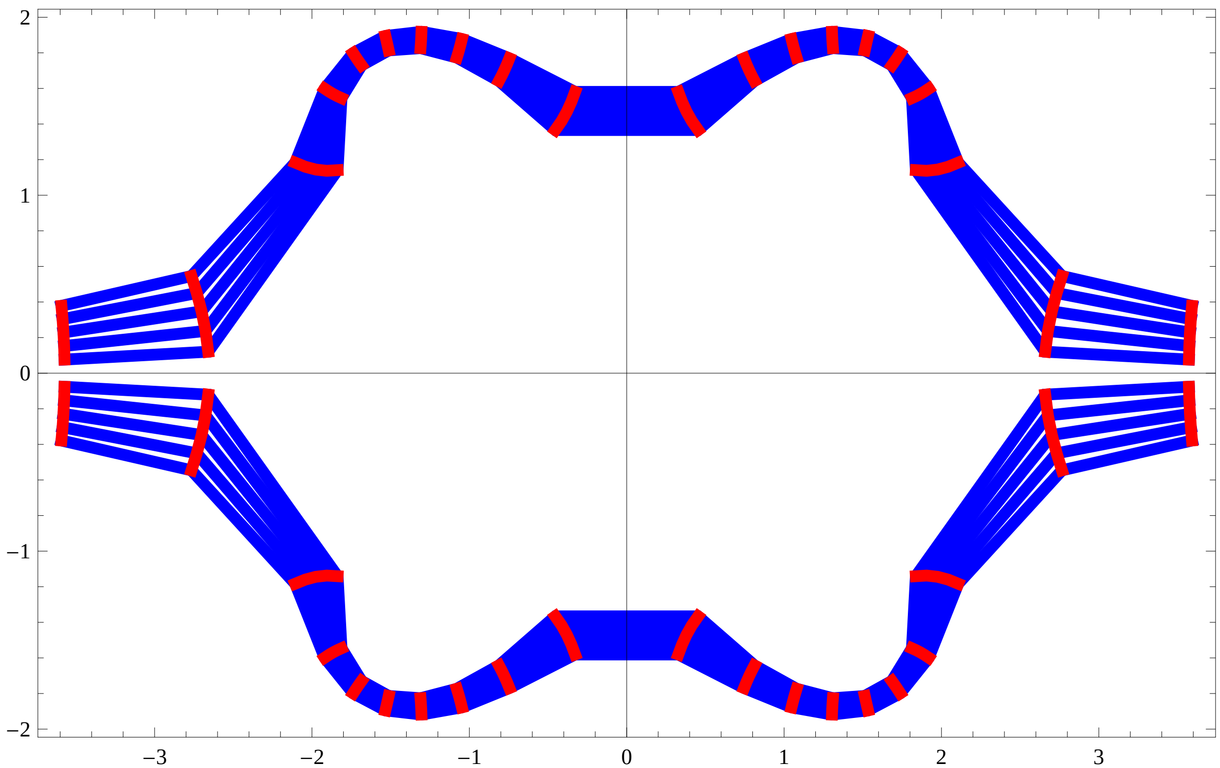


Figure 14: Complex values taken by λ^t when M^2 varies on horizontal lines close to the real axis in the M^2 plane.

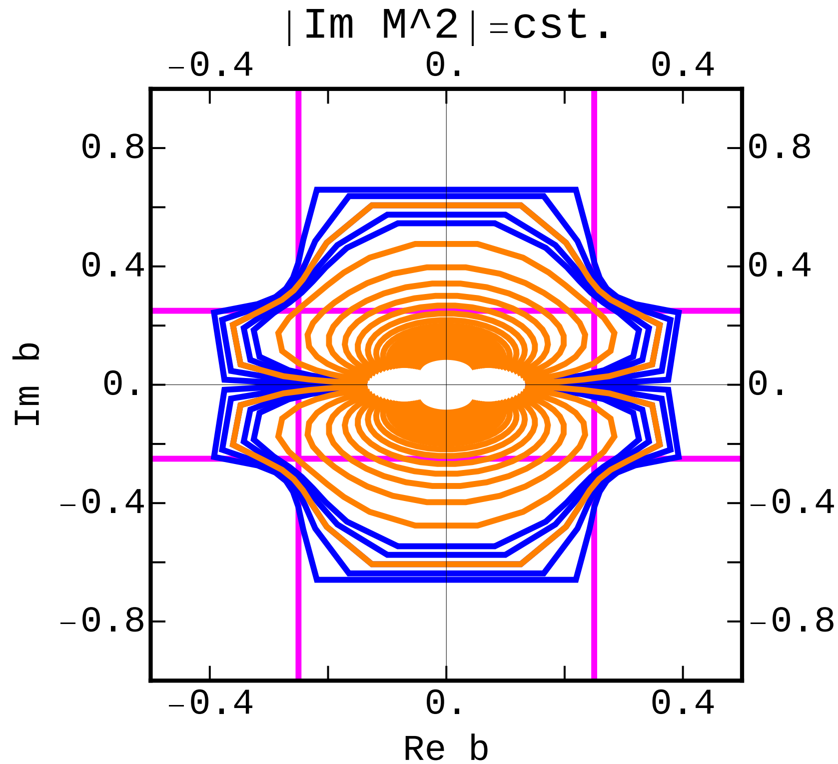


Figure 15: Complex values taken by $b = 1/\lambda^t$ when M^2 varies over the complex plane (here on horizontal lines in the M^2 plane; spacing 0.1 (blue) and 0.5 (turquoise)). Asymptotic limits are ± 0.25 and represent the logarithmic singularities at $M^2 = 0,\ 4$ and 8 (magenta).

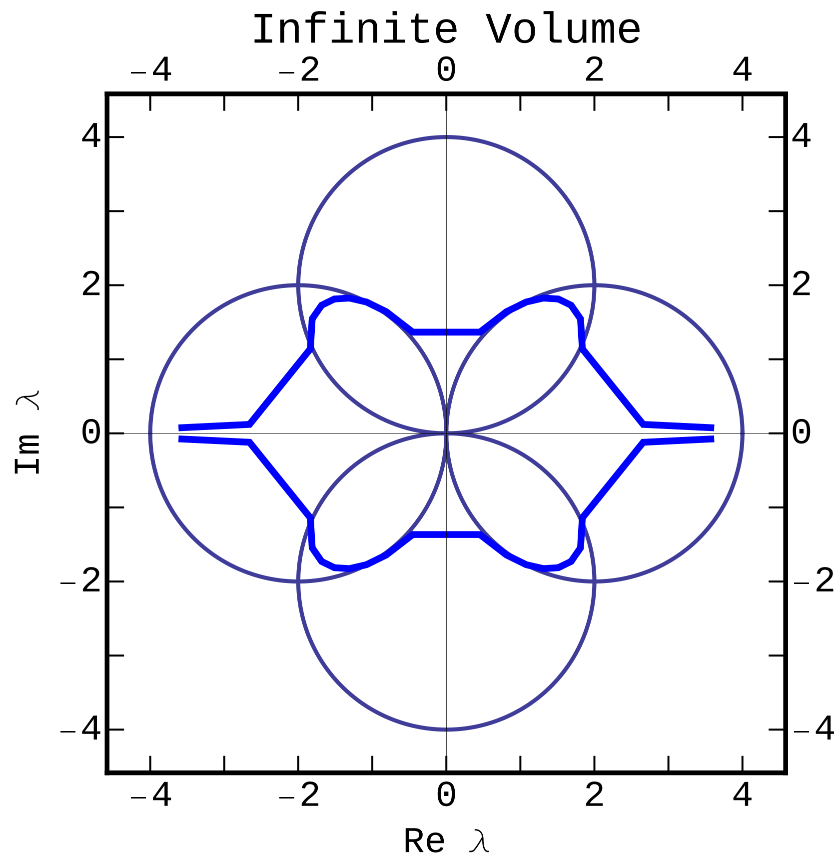


Figure 16: Complex values taken by λ^t when M^2 varies over the complex plane (here on horizontal and vertical lines in the M^2 plane); the circles are inverses of the asymptotic lines in the $1/\lambda^t$ plane.

Average energy

$\mathcal{E} =< E > /L^d = (1/2)(\lambda^t - M^2)$

Note that $0 \leq \mathcal{E} \leq 2D$ (the range is N-independent)

At large M^2, $M^2 \simeq \lambda^t$ so unlike the linear sigma model there is no cut at infinity.

Dispersion relations dominated by four-leaf clover path

Plausible scenario for LGT?

Density of states

$n(E) = \int \prod_x d^N\phi_x \delta(\vec{\phi}_x \vec{\phi}_x - 1)\delta(E[\{\phi\}] - E)$

$\delta(E[\{\phi\}] - E) = \int_{K-i\infty}^{K+i\infty} du \mathrm{e}^{u(E[\{\phi\}]-E)}$

Saddle point:

$\prod_{j=1}^{D} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \frac{1}{2\sum_{j=1}^{D}(1-\cos(k_j))+M^2} = u$

$M^2 = 1/u - 2\mathcal{E}$

These equations are equivalent to the previous ones except that now $\mathcal{E}$ is the independent variable and u is a function of $\mathcal{E}$ and plays the role of $1/\lambda^t$.

Entropy

$$\begin{aligned} f(\mathcal{E}) &= \ln(n(S))/(\mathcal{N}_p N) \\ &= (1/2)M^2 u + u\mathcal{E} \\ &\quad -(1/2)\log(u) - (1/2)\prod_{j=1}^{D}\int_{-\pi}^{\pi}\frac{dk_j}{2\pi}\ln(2\sum_{j=1}^{D}(1-\cos(k_j)) + M^2) \end{aligned}$$

$f(\mathcal{E}) = f(2D - \mathcal{E})$ (using s.p. equations)

small $\mathcal{E}$, $f(\mathcal{E}) \simeq (1/2)\ln(\mathcal{E})$

$\mathcal{E} \simeq D$, $f(\mathcal{E}) \simeq (-1/D)(\mathcal{E} - D)^2$

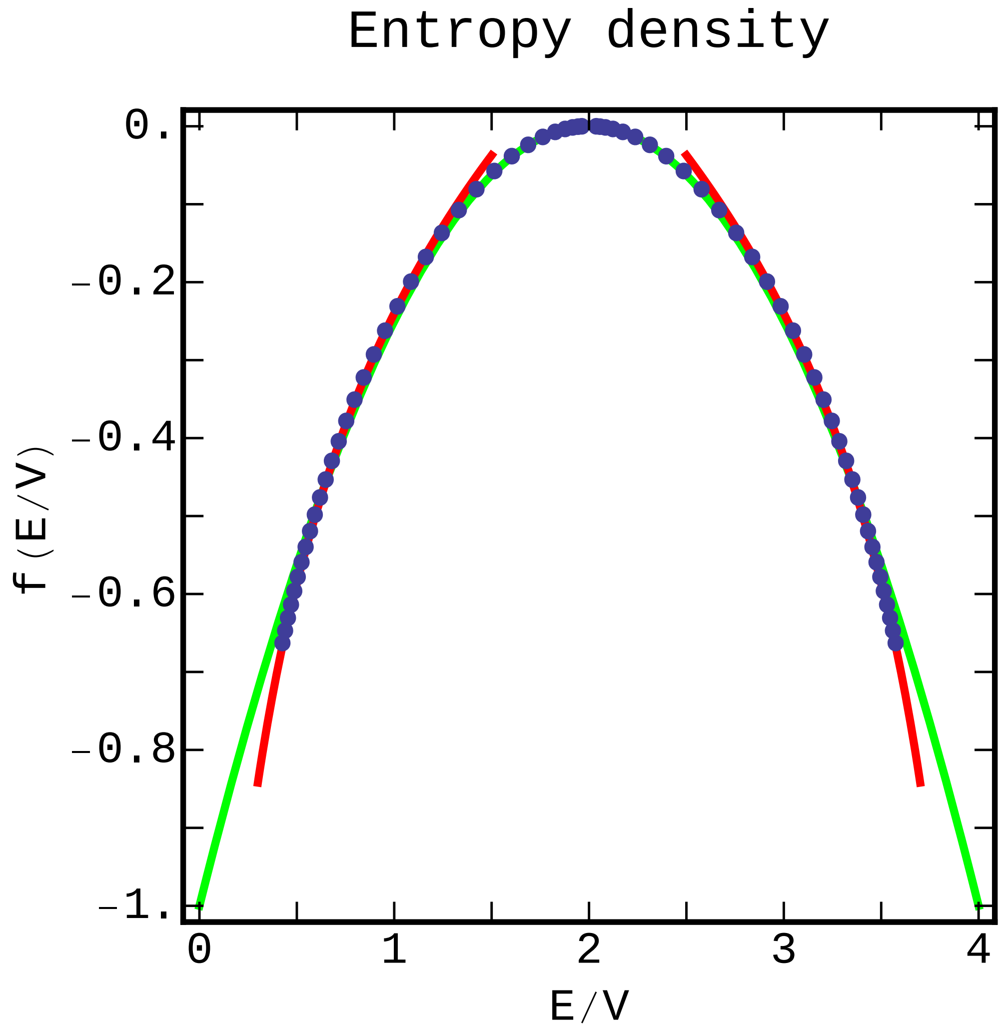


Figure 17: $f(\mathcal{E})$ and the leading weak and strong coupling expansions

Zeros of the partition functions

$$\begin{aligned}\oint_C db(dZ/db)/Z &= i2\pi \times \text{number of zeros in C} \\ &= V \oint_{C'} dM^2(db/dM^2)(1/2)(1/b - M^2)\end{aligned}$$

C' is the contour in the M^2 plane. If C' does not cross $[-8, 0]$, there are no zeros. So, there are no zeros of the partition function in the image of the cut M^2 plane. The apparent pole at $b = 0$ is cancelled by $M^2 \simeq 1/b$ in this limit. Consistent with preliminary numerical calculations using interpolations of $f(\mathcal{E})$ at infinite volume and large values of NV.

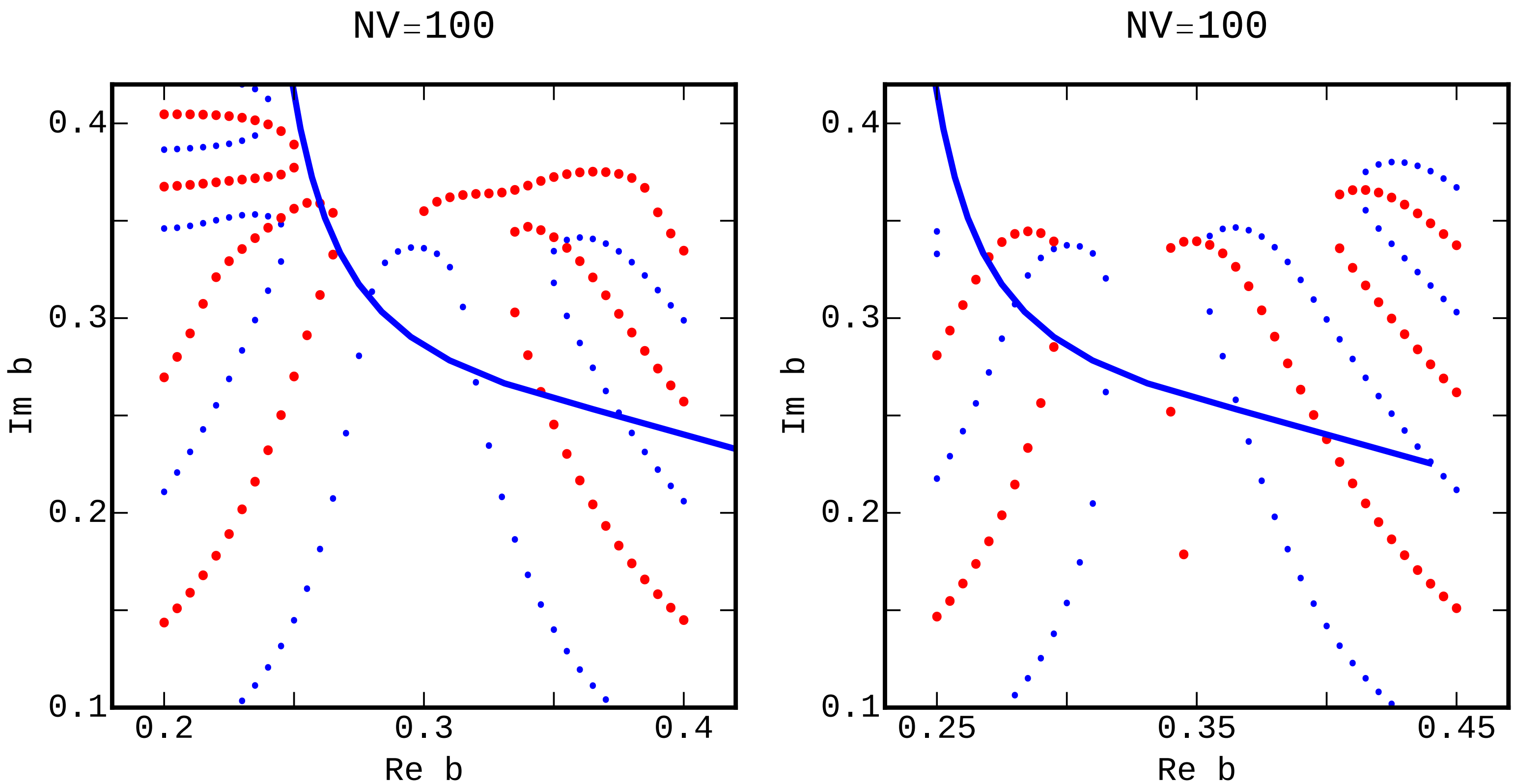


Figure 18: Preliminary search for zeros for $NV = 100$ with two different reweightings; zeros of Re (blue), zeros of Im (red); the solid blue line is the boundary of $b(M^2)$

Perspectives

- Large order in weak coupling expansions in LGT dominated by complex singularities of the density of states for $0 < |S| < S_{max}$. Non-perturbative effects in tail effects ($S \sim S_{max}$).

- "Hadamard" expansions (that includes $e^{-l\beta}$ effects) possible at least for reduced models.

- λ^t expansion sensitive to instability but not metastability

- Clover-leaf dispersion for LGT?

- Finite volume effects in progress (tomorrow's talk)