Factorization of Drell-Yan at Low q_T

TMDPDFs on-the-light-cone

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M.G. Echevarría, A. Idilbi, A. Schaefer, I. Scimemi, work in progress
MGE, AI, IS, {arXiv: 1111.4996}
MGE, Ahmad Idilbi, Ignazio Scimemi. Phys. Rev. D84 (2011) 011502. {arXiv:1104.0686}

Drell-Yan Scattering and the Structure of Hadrons
ECT, Trento, May 21-25, 2012*

Outline

★ Factorization Theorem for Drell-Yan at small q_T (using SCET):

First matching step: separation of modes

- ❖ Light-cone divergencies
- ❖ **Definition of TMDPDFs on-the-light-cone** (vs Collins)
  - Properties of the newly defined TMDPDF
    - **Free from Rapidity Divergencies**
    - **Universal**
    - **From TMDPDF to integrated PDF**

★ Factorization Theorem for Drell-Yan at small q_T:

Second matching step: OPE of the TMDPDF onto PDFs

- ❖ **Q^2 resummation *without* Collins-Soper evolution equation**
- ❖ **Evolution: Anomalous Dimension at 2-loop and 3-loop**
- ❖ **Resummed TMDPDF and its evolution**

★ Factorization Theorem for Drell-Yan at small q_T:

Final Factorization Theorem: Resummed Hadronic Tensor

★ Conclusions and Outlook

What Is SCET?

[Bauer, Fleming, Pirjol, Stewart '01, BPS '02]

- Soft-Collinear Effective Theory is an effective theory of QCD
- SCET describes interactions between low energy (u)soft and collinear fields (very energetic in one light-cone direction)
- Expand the lagrangian in powers of $\lambda \sim p_\perp / p_{col} \ll 1$
- SCET captures all the IR physics of QCD: matching is possible
- SCET is useful to prove factorization theorems and resum large logs

$$p_n^\mu = Q(1, \lambda^2, \lambda) \qquad \textit{n-collinear}$$
$$p_{\bar n}^\mu = Q(\lambda^2, 1, \lambda) \qquad \bar{n}\textit{-collinear}$$
$$p_{us}^\mu = Q(\lambda^2, \lambda^2, \lambda^2) \qquad \textit{ultrasoft (SCET-I)}$$
$$p_s^\mu = Q(\lambda, \lambda, \lambda) \qquad \textit{soft (SCET-II)}$$

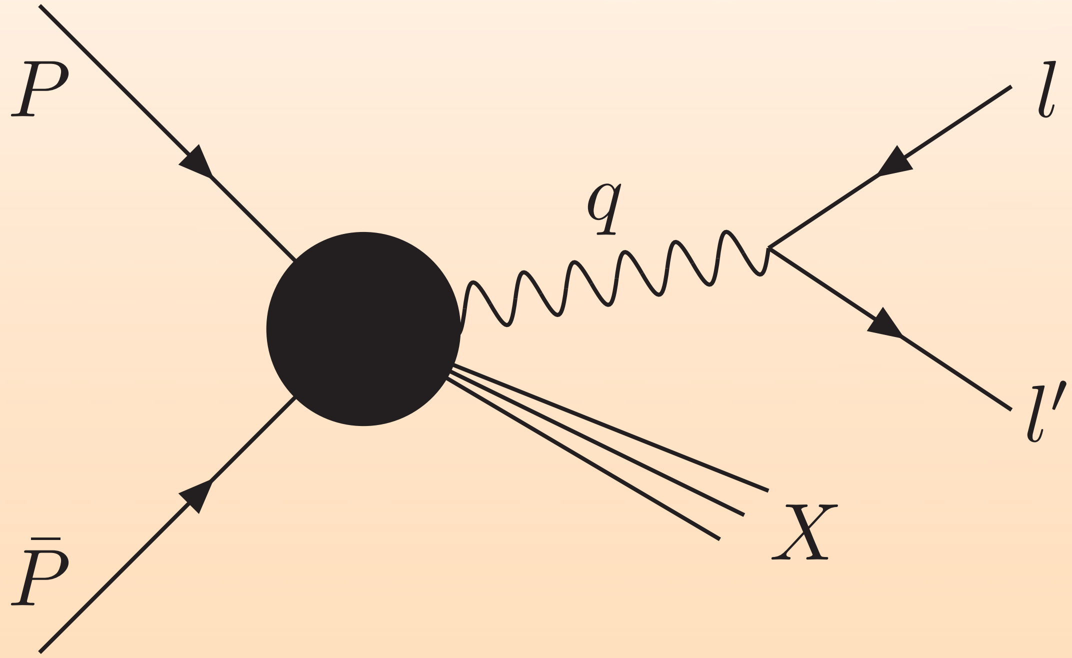
$$n^2 = \bar{n}^2 = 0\,, \quad n \cdot \bar{n} = 2$$
$$n^\mu = (1,0,0,1)$$
$$\bar{n}^\mu = (1,0,0,-1)$$

$$p^\mu = \bar{n}\cdot p \frac{n^\mu}{2} + n \cdot p \frac{\bar{n}^\mu}{2} + p_\perp^\mu$$
$$\equiv p_+^\mu + p_-^\mu + p_\perp^\mu \equiv (p^+, p^-, p_\perp)$$

➡ ***We use SCET to factorize Drell-Yan at small q_T and define the TMDPDFs.***

DY Factorization at Small q_T:

General Overview



$$q^2 = Q^2 \gg q_T^2 \gg \Lambda_{QCD}^2$$

$$\text{QCD} \longrightarrow \text{SCET-}q_T \longrightarrow \text{SCET-II}$$

$$\lambda \sim \frac{q_T}{Q} \qquad \lambda' \sim \frac{\Lambda_{QCD}}{Q}$$

$$\boxed{\tilde{M} \sim H(Q^2/\mu^2)\, \tilde{C}_{f/j}(b;Q,\mu)\, \tilde{C}_{\bar{f}/j'}(b;Q,\mu)\, f_{j/P}(\mu)\, f_{j'/\bar{P}}(\mu)}$$

Hard matching coefficient

Intermediate matching coefficients

PDFs

Q — H

$\downarrow \gamma_H$

q_T — $\tilde{C}_{f/j}\left(\tilde{C}_{\bar{f}/j'}\right)$

$\uparrow$ DGLAP

Λ_{QCD} — $f_{j/P}\left(f_{j'/\bar{P}}\right)$

DY Factorization at Small q_T:

QCD Matching onto SCET-q_T

$$d\Sigma = \frac{4\pi\alpha}{3q^2 s}\frac{d^4 q}{(2\pi)^4}\boxed{\frac{1}{4}\sum_{\sigma_1\sigma_2}\int d^4 y e^{-iqy}(-g_{\mu\nu})\langle N_1(P,\sigma_1)N_2(\bar{P},\sigma_2)|J^{\mu\dagger}(y)J^{\nu}(0)|N_1(P,\sigma_1)N_2(\bar{P},\sigma_2)\rangle}$$

QCD current → SCET-q_T current

$$J^\mu = \sum_q e_q \bar{\psi}\gamma^\mu\psi \longrightarrow J^\mu = C(Q^2/\mu^2)\sum_q e_q \bar{\xi}_{\bar{n}} W_{\bar{n}}^T S_{\bar{n}}^{T\dagger}\gamma^\mu S_n^T W_n^{T\dagger}\xi_n$$

- *(T stands for Transverse Wilson line: all matrix elements are gauge invariant!! [MGE, Idilbi, Scimemi '11])*
- Collinear, anti-collinear and soft fields decouple:

$$J_n(y) = \frac{1}{2}\sum_{\sigma_1}\langle N_1(P,\sigma_1)|\,\bar{\chi}_n(y)\,\frac{\bar{n}\!\!\!/}{2}\,\chi_n(0)\,|N_1(P,\sigma_1)\rangle|_{\text{zb subtracted}}$$

$$J_{\bar{n}}(y) = \frac{1}{2}\sum_{\sigma_2}\langle N_2(\bar{P},\sigma_2)|\,\bar{\chi}_{\bar{n}}(0)\,\frac{n\!\!\!/}{2}\,\chi_{\bar{n}}(y)\,|N_2(\bar{P},\sigma_2)\rangle_{\text{zb subtracted}}$$

$$S(y) = \langle 0|\,\mathrm{Tr}\,\bar{\mathbf{T}}\left[S_n^{T\dagger}S_{\bar{n}}^T\right](0^+,0^-,y_\perp)\mathbf{T}\left[S_{\bar{n}}^{T\dagger}S_n^T\right](0)\,|0\rangle$$

$$\chi_n = W_n^{T\dagger}\xi_n$$

DY Factorization at Small q_T:

Gauge Invariance: T-Wilson Line

- By the inclusion of the T-Wilson line we make the TMDPDF gauge invariant in regular and singular gauges.
- In [MGE, Idilbi, Scimemi '11] SCET formalism was extended to include singular gauges.

Light-Cone Gauge:

$$\bar{n}\cdot A_n = 0$$

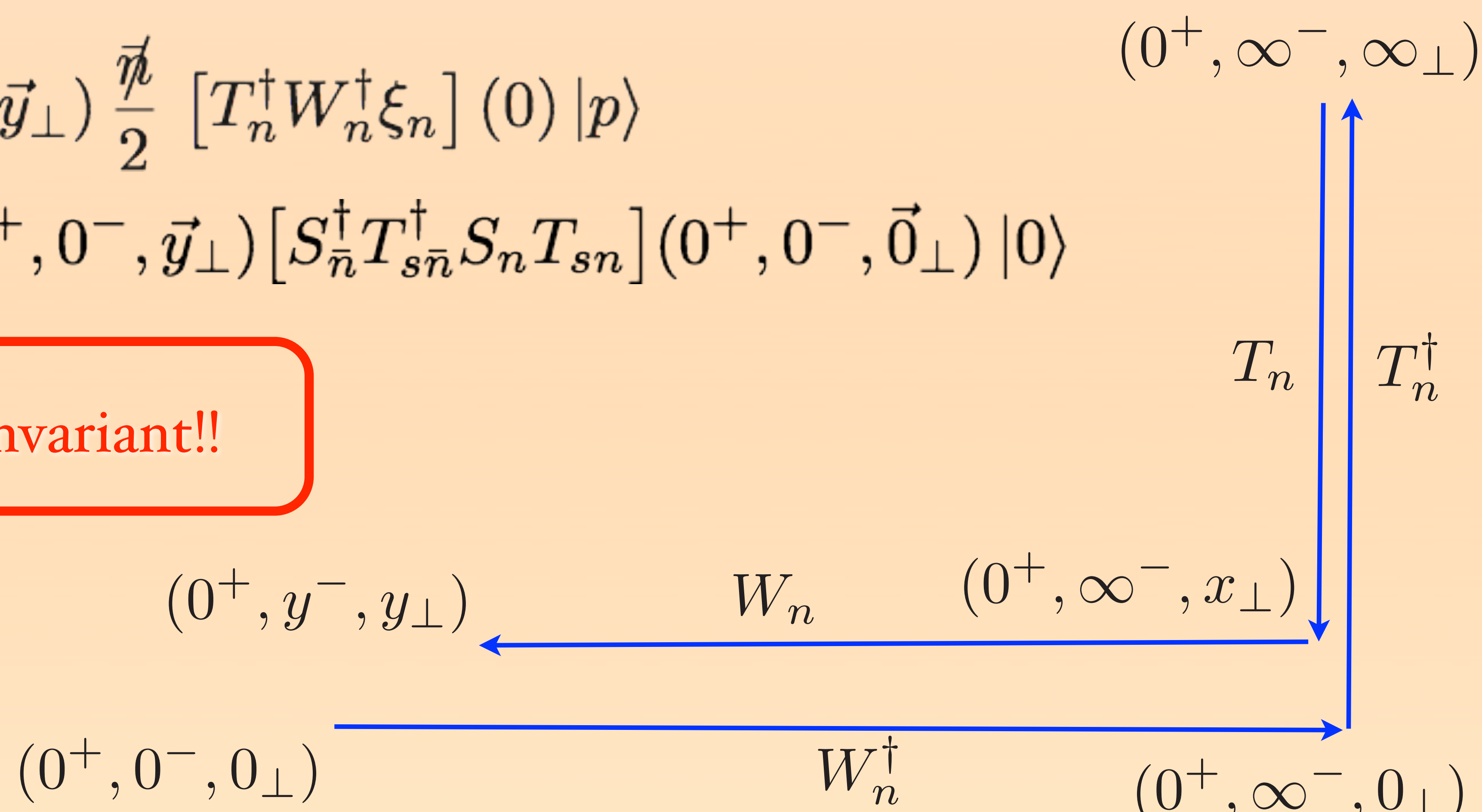
$$W_n = 1$$

$$T_n(x) = \bar{P}\exp\left[ig\int_{-\infty}^{0} d\tau \vec{l}_\perp \cdot \vec{A}_{n\perp}(x^+,\infty^-,\vec{x}_\perp + \vec{l}_\perp\tau)\right]$$

$$\tilde{\hat{J}}_n = \langle p|\ \left[\bar{\xi}_n W_n T_n\right](0^+,y^-,\vec{y}_\perp)\frac{\bar{\not{n}}}{2}\left[T_n^\dagger W_n^\dagger \xi_n\right](0)\,|p\rangle$$

$$\tilde{S} = \langle 0|\,\mathrm{Tr}\left[S_n^\dagger T_{sn}^\dagger S_{\bar{n}} T_{s\bar{n}}\right](0^+,0^-,\vec{y}_\perp)\left[S_{\bar{n}}^\dagger T_{s\bar{n}}^\dagger S_n T_{sn}\right](0^+,0^-,\vec{0}_\perp)\,|0\rangle$$

$$\boxed{\tilde{F}_{f/P} = \tilde{\hat{J}}_n \otimes \tilde{S}^{-\frac{1}{2}} \quad \text{is Gauge Invariant!!}}$$



DY Factorization at Small q_T:

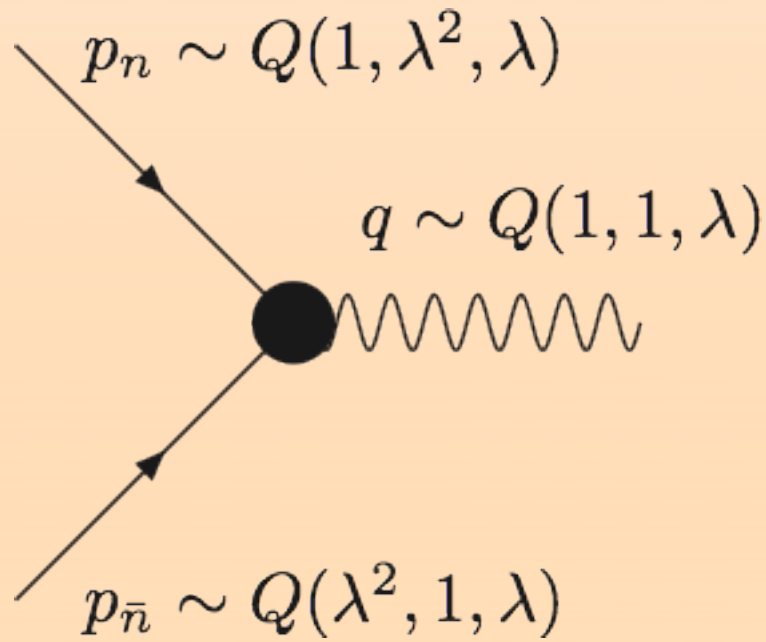
Taylor Expansion

$$\boxed{M = H(Q^2/\mu^2) \int d^4y\, e^{-iq\cdot y}\, J_n(y)\, J_{\bar{n}}(y)\, S(y)}$$

- This result includes subleading contributions (in powers of 1/Q2): Taylor Expansion.

The photon is hard:

$$\lambda \sim \frac{q_T}{Q}$$



$$y \sim \frac{1}{Q}(1, 1, \frac{1}{\lambda})$$

$$\left(\frac{\partial}{\partial y^-}, \frac{\partial}{\partial y^+}, \frac{\partial}{\partial y_\perp}\right) J_n(y) \sim Q(1, \lambda^2, \lambda)$$

$$\left(\frac{\partial}{\partial y^-}, \frac{\partial}{\partial y^+}, \frac{\partial}{\partial y_\perp}\right) J_{\bar{n}}(y) \sim Q(\lambda^2, 1, \lambda)$$

$$\left(\frac{\partial}{\partial y^-}, \frac{\partial}{\partial y^+}, \frac{\partial}{\partial y_\perp}\right) S(y) \sim Q(\lambda, \lambda, \lambda)$$

$$M = H(Q^2/\mu^2) \int d^4y\, e^{-iq\cdot y} \boxed{J_n(0^+, y^-, \vec{y}_\perp)\, J_{\bar{n}}(y^+, 0^-, \vec{y}_\perp)} \boxed{S(0^+, 0^-, \vec{y}_\perp)}$$

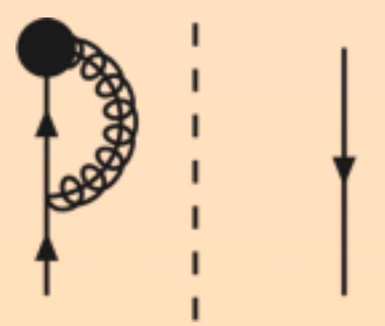
- [Ji, Ma, Yuan '05] introduced this soft function with dependence ONLY on the transverse component [contrary to Collins' earlier works] *"by hand"*. **We derive it!!**

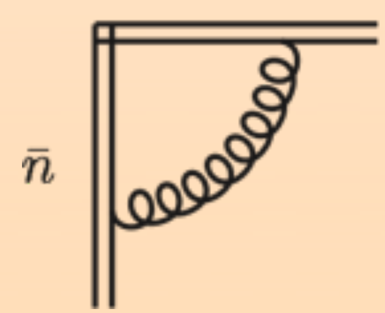
DY Factorization at Small q_T:

Double Counting

$$\boxed{M = H(Q^2/\mu^2) \int d^4y\, e^{-iq\cdot y}\, J_n(0^+, y^-, \vec{y}_\perp)\, J_{\bar{n}}(y^+, 0^-, \vec{y}_\perp)\, S(0^+, 0^-, \vec{y}_\perp)}$$

- Taking the soft limit of the contribution of the collinear Wilson line:





$$\mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{p^+ + k^+}{[k^+ - i\varepsilon][(p+k)^2 + i\varepsilon][k^2 + i\varepsilon]} \rightarrow \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^+ - i\varepsilon][k^- + i\varepsilon][k^2 + i\varepsilon]}$$

- Because there is a double counting issue... ***So we need to subtract the soft function!!***

$$\boxed{M = H(Q^2/\mu^2) \int d^4y\, e^{-iq\cdot y}\, \frac{\hat{J}_n(0^+, y^-, \vec{y}_\perp)}{S(0^+, 0^-, \vec{y}_\perp)}\, \frac{\hat{J}_{\bar{n}}(y^+, 0^-, \vec{y}_\perp)}{S(0^+, 0^-, \vec{y}_\perp)}\, S(0^+, 0^-, \vec{y}_\perp)}$$

DY Factorization at Small q_T: 1st Matching Step

Factorization of Modes

$$\boxed{M = H(Q^2/\mu^2) \int d^4y\, e^{-iq\cdot y}\, \frac{\hat{J}_n(0^+, y^-, \vec{y}_\perp)\, \hat{J}_{\bar{n}}(y^+, 0^-, \vec{y}_\perp)}{S(0^+, 0^-, \vec{y}_\perp)}}$$

We are not done yet!!

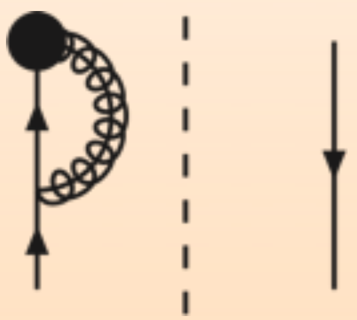
- This is **NOT** the final factorization theorem
- Up to now we have just separated the **modes**, not the long and short distance physics

- Now we **stop the derivation of the factorization theorem** to define the TMDPDF and show few of its properties.
- **Later we will continue** with the final steps towards the DY factorization theorem.

- Let's focus on one issue: **rapidity divergencies**
- It will lead us to the **definition of the TMDPDF**

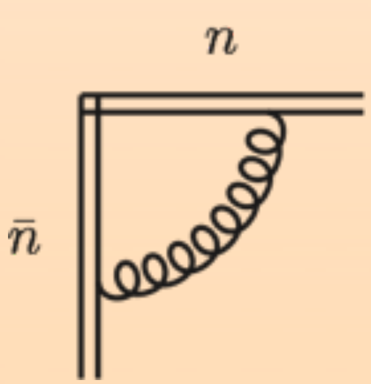
Light-cone (Rapidity) Divergencies

- Collinear and soft Wilson lines give mixed UV/rapidity divergencies (RD):



$$\hat{J}_{n1} = -2ig^2 C_F \delta(1-x)\delta^{(2)}(\vec{k}_{n\perp}) \times \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{p^+ + k^+}{[k^+ - i\varepsilon][(p+k)^2 + i\varepsilon][k^2 + i\varepsilon]}$$

$$\boxed{\frac{1}{\varepsilon_{\rm UV}} \int_0^1 dt \frac{1}{t}}$$



$$S_1 = -2ig^2 C_F \delta^{(2)}(\vec{k}_{n\perp}) \times \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^+ - i\varepsilon][k^- + i\varepsilon][k^2 + i\varepsilon]}$$

$$\boxed{(+2) \times \frac{1}{\varepsilon_{\rm UV}} \int_0^1 dt \frac{1}{t}}$$

- RD do NOT cancel even when we combine virtual and real diagrams.
- For PDF we have similar light-cone singularities, but they cancel between virtual and real.
- The RD coming from the soft function is double the one coming from the collinear (this is due to the double counting issue...).

Now...

- In QCD, the hadronic tensor M (partonic) is free from mixed divergencies.
- And actually it has even no rapidity divergencies

Operator Definition of the TMDPDF

- In order to cancel the mixed divergencies we define the TMDPDF as:

$$\boxed{F_{f/P}(x,\vec{k}_{n\perp}) = \frac{1}{2}\int \frac{dr^- d^2\vec{r}_\perp}{(2\pi)^3} e^{-i(\frac{1}{2}r^- xp^+ - \vec{r}_\perp\cdot\vec{k}_{n\perp})} \frac{\hat{J}_n(0^+,r^-,\vec{r}_\perp)}{\sqrt{S(0^+,0^-,\vec{r}_\perp)}}}$$

- With these definition we get (in impact parameter space):

$$\tilde{M} = H(Q^2/\mu^2)\,\tilde{F}_{f/P}(x_1,b;Q^2,\mu)\,\tilde{F}_{\bar{f}/\bar{P}}(x_2,b;Q^2,\mu)$$

No soft function in the factorization theorem!! (Agreement with [Collins '11])

- Seven definitions of TMDPDF "in the market":
  - Collins '82: *just collinear (off-the-LC)*
  - Ji, Ma, Yuan '05: *collinear with subtraction of complete soft function (off-the-LC)*
  - Cherednikov, Stefanis '08: *collinear with subtraction of complete soft function (LC gauge)*
  - Mantry, Petriello '10: *fully unintegrated collinear matrix element*
  - Becher, Neubert '11: *there is no definition of TMDPDF*
  - Collins '11: *collinear with subtraction of square root of soft function (off-the-LC "strange")*
  - Chiu, Jain, Neill, Rothstein '12: *collinear matrix element (Rapidity Regulator)*

Free from Rapidity Divergencies

- Collinear and soft matrix elements have un-regularized divergencies.
- Old idea by Collins and Soper: go off-the-light-cone.

- But we choose a different path: stay on-the-light-cone and use another regulator
- This regulator does not distinguish between IR and LC divergencies
- *All the properties of the TMDPDF are regulator-independent, of course!!*

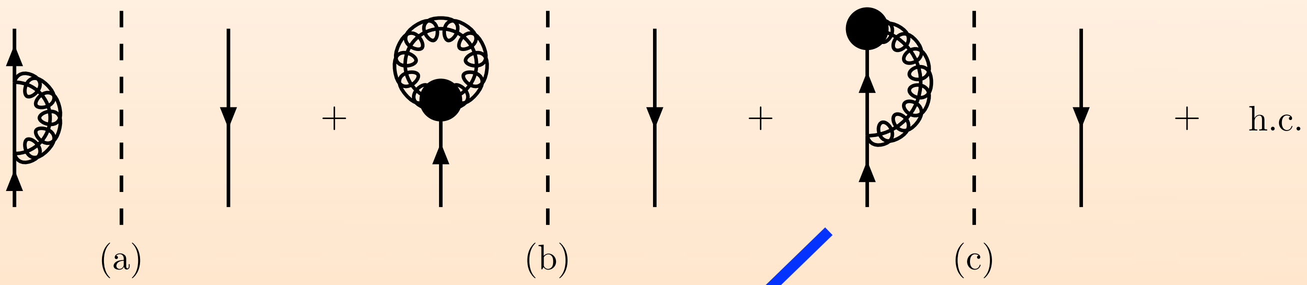
δ-Regulator

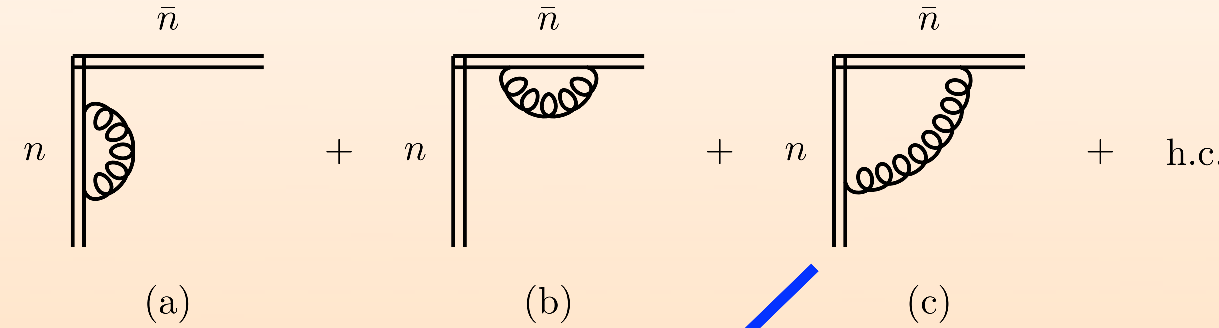
$$\frac{i(\not{p}+\not{k})}{(p+k)^2+i\Delta^-} \longrightarrow \frac{1}{k^- \pm i\delta^-}\,,\ \delta^- = \frac{\Delta^-}{p^+}$$

[Chiu, Fuhrer, Hoang, Kelley, Manohar '09]

Relation between regulators in propagators and Wilson lines

Results at One Loop (Virtual)





$$\frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta^{(2)}(\vec{k}_{n\perp})\left[\boxed{\frac{2}{\varepsilon_{\rm UV}}\ln\frac{\Delta}{Q^2}}+\frac{2}{\varepsilon_{\rm UV}}\right.$$
$$\left.-\ln^2\frac{\Delta^2}{Q^2\mu^2}-2\ln\frac{\Delta}{\mu^2}+\ln^2\frac{\Delta}{\mu^2}+2-\frac{7\pi^2}{12}\right]$$

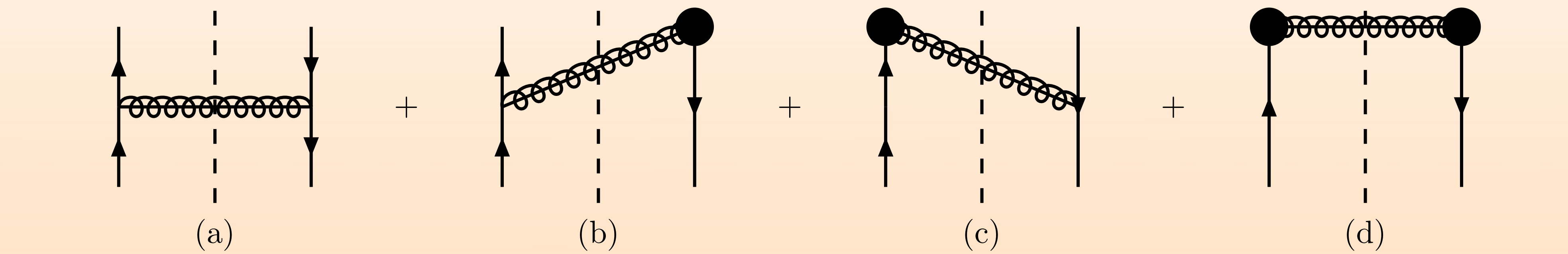
$$-\frac{\alpha_s C_F}{2\pi}\delta^{(2)}(\vec{k}_{n\perp})\left[\frac{2}{\varepsilon^2_{\rm UV}}-\boxed{\frac{2}{\varepsilon_{\rm UV}}\ln\frac{\Delta^2}{Q^2\mu^2}}\right.$$
$$\left.+\ln^2\frac{\Delta^2}{Q^2\mu^2}+\frac{\pi^2}{2}\right]$$

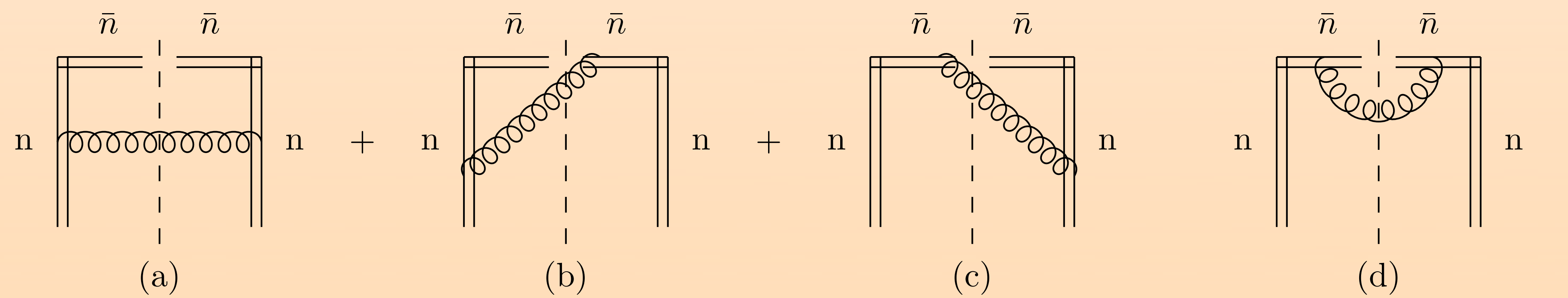
- Subtracting the square root of the soft function to the collinear:

$$F^v_{f/P,1}=\frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta^{(2)}(\vec{k}_T)\left[\frac{1}{\varepsilon^2_{\rm UV}}+\frac{1}{\varepsilon_{\rm UV}}\left(\frac{3}{2}+\ln\frac{\mu^2}{Q^2}\right)\right.$$
$$\left.-\frac{3}{2}\ln\frac{\Delta}{\mu^2}-\frac{1}{2}\ln^2\frac{\Delta^2}{Q^2\mu^2}+\ln^2\frac{\Delta}{\mu^2}+\frac{7}{4}-\frac{\pi^2}{3}\right]$$

Free from mixed divergencies!!

Results at One Loop (Real)





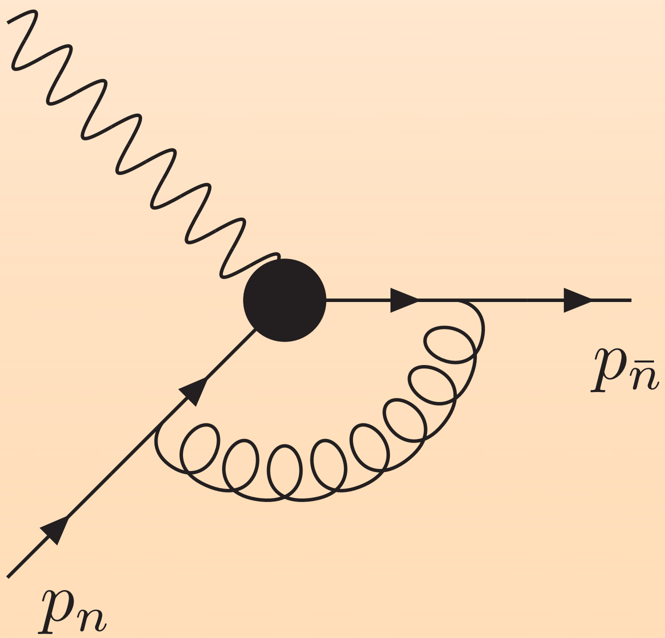
- Same story for reals: subtracting the square root of the soft function to the collinear, the rapidity divergencies cancel.

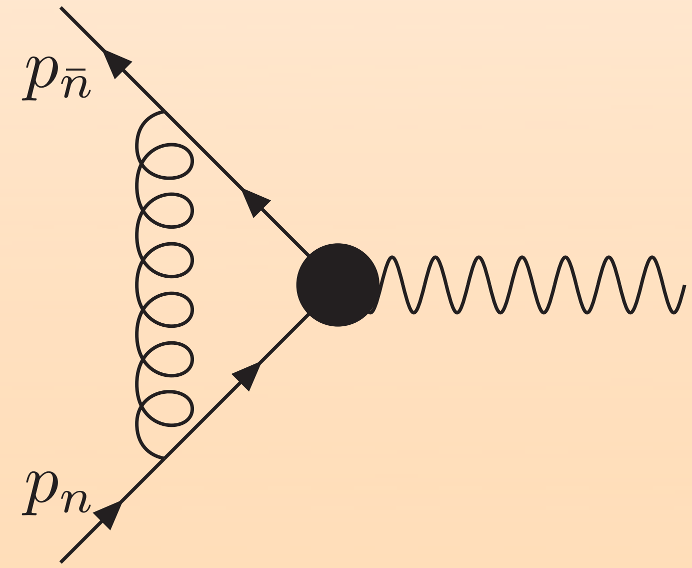
$$F^{r}_{f/P,1} = \frac{2\alpha_s C_F}{(2\pi)^{2-2\varepsilon}} \frac{1}{k_T^2} \left[(1-\varepsilon)(1-x) + \frac{2x}{(1-x)_+} + \delta(1-x) \ln \frac{Q^2}{k_T^2} \right]$$

The real part is independent of Δ-regulator!!

Universality of the TMDPDF

- Are the TMDPDFs for DIS and DY kinematics the same?
- Universality is needed to maintain the predictive power of perturbative QCD.





- Taking n-collinear and soft limits we get the following Wilson lines:

$$\tilde{W}_n(x) = \bar{P}\exp\left[-ig\int_0^\infty ds\,\bar{n}\cdot A_n(x+\bar{n}s)\right] \qquad W_n(x) = \bar{P}\exp\left[ig\int_{-\infty}^0 ds\,\bar{n}\cdot A_n(x+s\bar{n})\right]$$

$$\tilde{S}_{\bar{n}}(x) = P\exp\left[-ig\int_0^\infty ds\,\bar{n}\cdot A_s(x+\bar{n}s)\right] \qquad S_{\bar{n}}(x) = \bar{P}\exp\left[ig\int_{-\infty}^0 ds\,\bar{n}\cdot A_s(x+s\bar{n})\right]$$

- We have different Wilson lines for different kinematics.

Universality of the Soft Function

- Example: soft function for DY and DIS kinematics (virtual and real contributions)

$$S_1^{v,DY} = -2ig^2 C_F \delta^{(2)}(\vec{k}_{n\perp})\mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\boxed{[k^+ - i\delta^+]}[k^- + i\delta^-][k^2 + i0]} + h.c.$$

$$S_1^{v,DIS} = -2ig^2 C_F \delta^{(2)}(\vec{k}_{n\perp})\mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\boxed{[k^+ + i\delta^+]}[k^- + i\delta^-][k^2 + i0]} + h.c.$$

$$S_1^{r,DY} = -4\pi g^2 C_F \int \frac{d^dk}{(2\pi)^d}\delta^{(2)}(\vec{k}_\perp + \vec{k}_{n\perp})\delta(k^2)\theta(k^+)\frac{1}{\boxed{[k^+ + i\delta^+]}[-k^- + i\delta^-]} + h.c.$$

$$S_1^{r,DIS} = -4\pi g^2 C_F \int \frac{d^dk}{(2\pi)^d}\delta^{(2)}(\vec{k}_\perp + \vec{k}_{n\perp})\delta(k^2)\theta(k^+)\frac{1}{\boxed{[k^+ - i\delta^+]}[-k^- + i\delta^-]} + h.c.$$

$$\boxed{S_1^{v,DIS} = S_1^{v,DY} + \frac{\alpha_s C_F}{2\pi}\delta^{(2)}(\vec{k}_T)\,\pi^2} \qquad \boxed{S_1^{r,DIS} = S_1^{r,DY} - \frac{\alpha_s C_F}{2\pi}\delta^{(2)}(\vec{k}_T)\,\pi^2}$$

- **The soft function is universal!!**

Universality of the Collinear

• For the naive collinear we get something similar:

$$\boxed{\hat{J}_{n1}^{v,DIS} = \hat{J}_{\bar{n}1}^{v,DY} + \frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta^{(2)}(\vec{k}_{nT})\,\pi^2}$$

$$\boxed{\hat{J}_{n1}^{r,DIS} = \hat{J}_{\bar{n}1}^{r,DY} - \frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta^{(2)}(\vec{k}_{T})\,\pi^2}$$

• It turns out that **both naive collinear and soft matrix elements are universal** (then also the pure collinear, as expected)

So the TMDPDF is UNIVERSAL!!

• **Interestingly enough this result has never been established explicitly before.**

From TMDPDF to PDF

- Can we recover the PDF from the TMDPDF by simple integration?
- Up to now this turned out to be elusive: [Ji, Ma, Yuan '04], [Cherednikov, Stefanis '08-'11]

- Taking the TMDPDF calculated in pure Dim. Reg. and integrating it:

$$\mu^{2\varepsilon}\int d^{2-2\varepsilon}\vec{k}_T\, F^{v}_{f/P}(x,k_T;Q^2,\mu) =$$
$$\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[\frac{1}{\varepsilon^2_{\rm UV}}+\frac{1}{\varepsilon_{\rm UV}}\left(\frac{3}{2}+\ln\frac{\mu^2}{Q^2}\right)-\frac{1}{\varepsilon^2_{\rm IR}}-\frac{1}{\varepsilon_{\rm IR}}\left(\frac{3}{2}+\ln\frac{\mu^2}{Q^2}\right)\right]$$
$$\mu^{2\varepsilon}\int d^{2-2\varepsilon}\vec{k}_T\, F^{r}_{f/P}(x,k_T;Q^2,\mu) =$$
$$\frac{\alpha_s C_F}{2\pi}\left\{\left[\delta(1-x)\ln\frac{Q^2}{\mu^2}+\mathcal{P}_{q/q}-\frac{3}{2}\delta(1-x)\right]\left(\frac{1}{\varepsilon_{\rm UV}}-\frac{1}{\varepsilon_{\rm IR}}\right)-\delta(1-x)\left(\frac{1}{\varepsilon^2_{\rm UV}}-\frac{1}{\varepsilon^2_{\rm IR}}\right)\right\}$$

$$\boxed{\mu^{2\varepsilon}\int d^{2-2\varepsilon}\vec{k}_T\, F_{f/P}(x,k_T;Q^2,\mu) = \delta(1-x)+\frac{\alpha_s C_F}{2\pi}\mathcal{P}_{q/q}\left(\frac{1}{\varepsilon_{\rm UV}}-\frac{1}{\varepsilon_{\rm IR}}\right) = f_{q/P}(x;\mu)}$$

- We just need to regulate the TMDPDF and the PDF in the same way to get it.
- Even including the soft function in the TMDPDF we still can recover the PDF

DY Factorization at Small q_T: 2nd Matching Step

OPE of TMDPDF onto PDFs

- The TMDPDF has perturbative content when q_T is perturbative.
- We can do an OPE of the TMDPDF onto the PDFs in impact parameter space, integrating out the intermediate scale q_T. ***Second matching step!!***

$$\boxed{\tilde{F}_{f/P}(x,b;Q^2,\mu) = \sum_{j=q,g} \int_x^1 \frac{dx'}{x'} \tilde{C}_{f/j}\left(\frac{x}{x'},b;Q^2,\mu\right) f_{j/P}(x';\mu) + \mathcal{O}\left((\Lambda_{QCD} b)^a\right)}$$

- Properties of the coefficients $C_{f/j}$:
  - Independent of the IR regulator (as any matching coefficient!!)
  - It is supposed to live at the intermediate scale q_T. All logs should be cancelled by one choice for the intermediate scale (like at threshold).

  ➡ **BUT: it has a subtle Q^2-dependence**

$$\tilde{C}_{f/q}(x,b;Q^2,\mu) = \delta(1-x) + \frac{\alpha_s C_F}{2\pi}\Big[-\mathcal{P}_{q/q} L_T + (1-x) \\ -\delta(1-x)\left(\frac{1}{2}L_T^2 - \frac{3}{2}L_T + \ln\frac{Q^2}{\mu^2}L_T + \frac{\pi^2}{12}\right)\Big]$$

$$L_T = \ln\frac{\mu^2 b^2}{4e^{-2\gamma_E}}$$

- Accidentally, $\ln(Q^2/\mu^2)$ can be eliminated by choosing μ, but NOT at higher orders.
- Logs cannot be combined in one single log, like at threshold: ***Resummation is needed!!***

Q^2-Resummation (I)

- From the definition of the TMDPDF:

$$\ln \tilde{F}_{f/P} = \ln \tilde{\hat{J}}_n - \frac{1}{2} \ln \tilde{S}$$

- Using the Δ-regulator, Lorentz invariance and dimensional analysis:

$$\ln \tilde{\hat{J}}_n = \mathcal{R}_F \left(x; \alpha_s, L_T, \ln \frac{\delta^+}{p^+} = \ln \frac{\Delta}{Q^2} \right)$$
$$\ln \tilde{S} = \mathcal{R}_S \left(\alpha_s, L_T, \ln \frac{\delta^+ \delta^-}{\mu^2} = \ln \frac{\Delta^2}{Q^2 \mu^2} \right)$$

- Since the TMDPDF (Wilson coefficients and PDFs) is **free from rapidity divergencies** to all orders in perturbation theory:

$$\boxed{\frac{d \ln \tilde{F}_{f/P}}{d \ln \Delta} = 0}$$

This is analogous to Collins-Soper evolution equation!!

Q^2-Resummation (II): *"Linearity"*

- The last equation implies that the naive collinear and the soft are linear in their last arguments, so we get the linearity in the $\ln(Q^2/\mu^2)$:

$$\boxed{\ln\tilde{F}_{f/P} = \ln\tilde{F}^{\not Q}_{f/P} - D(b;\mu)\ln\frac{Q^2}{\mu^2}}$$

Independent of Q^2!!

Independent of Q^2!!

$$\boxed{\tilde{C}_{f/j}(x,b;Q^2,\mu) = \left(\frac{Q^2}{\mu^2}\right)^{-D(b;\mu)} \tilde{C}^{\not Q}_{f/j}(x,b;\mu)}$$

- Applying RGE to the hadronic tensor we get:

$$\boxed{\frac{dD(b;\mu)}{d\ln\mu} = \Gamma_{cusp}(\alpha_s)}$$

- The Q^2-factor is extracted for each TMDPDF individually.
- **We do not need Collins-Soper evolution equation to resum the logs of Q^2.**
- We know cusp AD at 3-loops, **so we know D at 2-loops!!**

Evolution: Anomalous Dimension

- Applying the RGE to M we get the AD of the TMDPDF:

$$\tilde{M} = H(Q^2/\mu^2)\, \tilde{F}_{f/P}(x_1, b; Q^2, \mu)\, \tilde{F}_{\bar{f}/\bar{P}}(x_2, b; Q^2, \mu)$$

$$\frac{d\ln\tilde{M}}{d\ln\mu} = 0 = \gamma_H + \gamma_n + \gamma_{\bar{n}} \qquad \gamma_H = A(\alpha_s)\ln\frac{Q^2}{\mu^2} + B(\alpha_s)$$

$$\boxed{\gamma_n = \gamma_{\bar{n}} = -\frac{1}{2}\gamma_H} \qquad \tilde{F}_{f/P}\left(x, b; Q^2, \mu_f\right) = \exp\left\{\int_{\mu_i}^{\mu_f} \frac{d\mu'}{\mu'}\gamma_n\left(\alpha_s(\mu), \ln\frac{Q^2}{\mu'^2}\right)\right\}\tilde{F}_{f/P}\left(x, b; Q^2, \mu_i\right)$$

$$\boxed{\begin{aligned}\gamma_{n2} = -\frac{1}{2}\gamma_{H2} = -\frac{1}{2}2\left(\frac{\alpha_s}{\pi}\right)^2\Bigg\{&\left[\left(\frac{67}{36} - \frac{\pi^2}{12}\right)C_A - \frac{5}{18}N_f\right]C_F\ln\frac{Q^2}{\mu^2} \\ &+ \left(\frac{13}{4}\zeta(3) - \frac{961}{16\times 27} - \frac{11}{48}\pi^2\right)C_A C_F + \left(\frac{\pi^2}{24} + \frac{65}{8\times 27}\right)N_f C_F + \left(\frac{\pi^2}{4} - \frac{3}{16} - 3\zeta(3)\right)C_F^2\Bigg\}\end{aligned}}$$

- A and B are known up to three loops!!
- **We have by free the AD of the TMDPDF at 3-loops!!**
- Collins' AD can only be calculated at 1-loop, since there is no relation with the AD of the hard matching coefficient.

Resummed TMDPDF on-the-LC!!

$$\tilde{F}_{f/P}\left(x,b;Q^2,\mu=Q\right)=\overbrace{\sum_{j=q,g}\int_x^1\frac{dx'}{x'}\,\tilde{C}^{Q}_{f/j}\left(\frac{x}{x'},b^*;\mu_{b^*}\right)\,f_{j/P}(x';\mu_{b^*})}^{\text{A}}$$

$$\times\overbrace{\exp\left\{-D\left(b^*,\mu_{b^*}\right)\ln\frac{Q^2}{\mu_{b^*}^2}+\int_{\mu_{b^*}}^{\mu=Q}\frac{d\mu'}{\mu'}\left[\gamma_n\left(\alpha_s(\mu');\ln\frac{Q^2}{\mu'^2}\right)\right]\right\}}^{\text{B}}\times\overbrace{\tilde{F}^{NP}_{f/P}\left(x,b;\mu=Q,Q_0\right)}^{\text{C}}$$

$$\mu_b=2e^{-\gamma_E}/b \qquad\qquad b^*(b)=b/\sqrt{1+b^2/b_{max}^2}$$

- We need to set a cutoff in b to avoid the Landau pole and separate perturbative from non-perturbative (NP) contributions.
- Match the TMDPDF onto PDFs for large k_T (small b).
- Use a NP model for small k_T (large b), for example BLNY.
- Running in μ and exponentiation of logs of Q^2. No Collins-Soper!!
- ***There are no un-physical or extra parameters like the rapidity of CS!!***

Order	Accuracy $\sim\alpha_s^n L^k$	γ_q	γ_K	$\tilde{C}_n$	D	
LL	$n+1\le k\le 2n$ (α_s^{-1})	tree	1-loop	tree	tree	
NLL (LO)	$n\le k\le 2n$ (α_s^0)	1-loop	2-loop	tree	1-loop	Previous
NNLL (NLO)	$n-1\le k\le 2n$ (α_s)	2-loop	3-loop	1-loop	2-loop	Future
NNNLL (NNLO)	$n-2\le k\le 2n$ (α_s^2)	3-loop	4-loop	2-loop	3-loop	

Evolution of the TMDPDF

$$\tilde{F}_{f/P}(x,b;Q_f^2,\mu_f=Q_f)=\tilde{F}_{f/P}(x,b;Q_i^2,\mu_i=Q_i)$$
$$\times\exp\left\{\boxed{-D(b,\mu_i)}\ln\frac{Q_f^2}{Q_i^2}+\int_{\mu_i=Q_i}^{\mu_f=Q_f}\frac{d\mu'}{\mu'}\boxed{\gamma_n\left(\alpha_s(\mu'),\ln\frac{Q_f^2}{\mu'^2}\right)}\right\}\exp\left\{g_K(b)\ln\frac{Q_f}{Q_i}\right\}$$

$$\tilde{F}_{f/P}^{Collins}(x,b;\zeta_{F,f},\mu_f=Q_f)=\tilde{F}_{f/P}^{Collins}(x,b;\zeta_{F,i},\mu_i=Q_i)$$
$$\times\exp\left\{\boxed{\tilde{K}(b,\mu_i)}\ln\frac{\sqrt{\zeta_{F,f}}}{\sqrt{\zeta_{F,i}}}+\int_{\mu_i=Q_i}^{\mu_f=Q_f}\frac{d\mu'}{\mu'}\boxed{\gamma_F\left(\alpha_s(\mu'),\ln\frac{\zeta_{F,f}}{\mu'^2}\right)}\right\}\exp\left\{g_K(b)\ln\frac{\sqrt{\zeta_{F,f}}}{\sqrt{\zeta_{F,i}}}\right\}$$

- **Once one makes an ANSATZ over the values of zetas and relate them to Qs, Collins' evolution is the same as ours.**
- The evolution of our TMDPDF is done entirely on-the-LC with just the hard probe Q.
- We do not need to evolve with Collins-Soper in any un-physical parameter like zeta.
- The evolution has a perturbative part and a non-perturbative part, because the NP model is scale-dependent.
- **Our formalism is much more simple: on-the-light-cone!!**
- **Compatible with factorization theorem: we can resum logs for DY on-the-LC and without Collins-Soper, just through the "linearity" argument.**
- **We can go to higher orders in perturbation theory (3-loop AD by free!!)**.

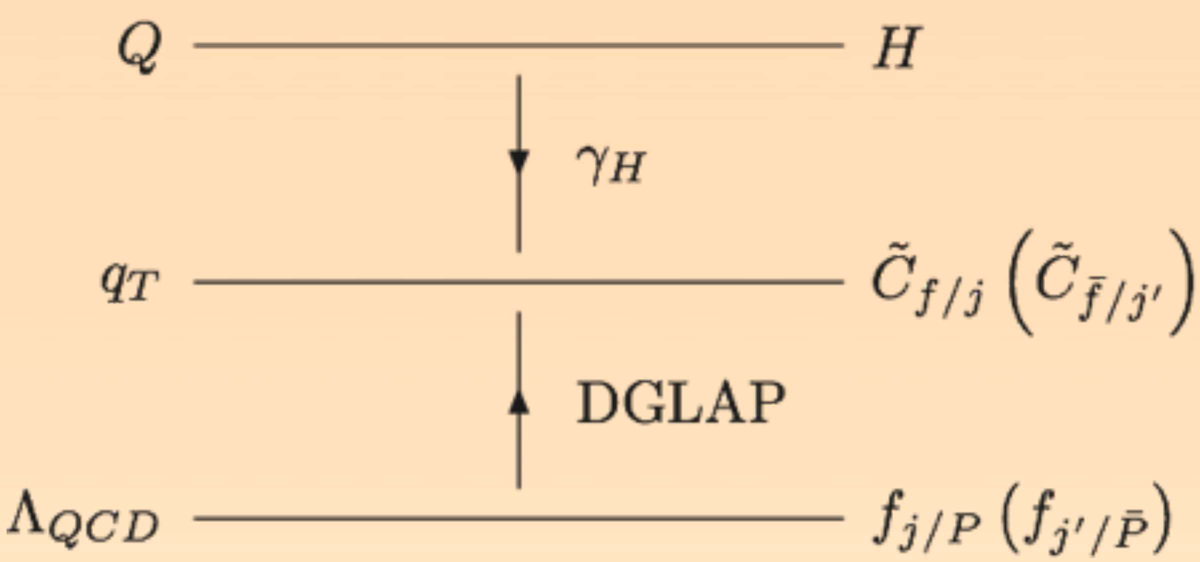
DY Factorization at Small q_T: Final Result

Resummed Hadronic Tensor

- Finally, we get the factorization theorem for DY at low q_T:

$$M = \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{b}_\perp} \int_x^1 \frac{dx'}{x'} \int_z^1 \frac{dz'}{z'} \sum_f e_f^2 \sum_{j=q,g} \sum_{j'=\bar{q},g} \boxed{f_{j/P}(x';\mu_{b^*})\, f_{j'/\bar{P}}(z';\mu_{b^*})}\, \boxed{\tilde{M}^{NP}(x,z,b;Q,Q_0)}$$
$$\times \boxed{H(Q^2/Q^2)\exp\left\{\int_Q^{\mu_{b^*}} \frac{d\mu'}{\mu'}\gamma_H\right\}}\, \boxed{\exp\left\{-2D(b^*,\mu_{b^*})\ln\frac{Q^2}{\mu_{b^*}^2}\right\} \tilde{C}^{\mathcal{Q}}_{f/j}\left(\frac{x}{x'},b^*;\mu_{b^*}\right) \tilde{C}^{\mathcal{Q}}_{\bar{f}/j'}\left(\frac{z}{z'},b^*;\mu_{b^*}\right)}$$

- Short and long distance physics are finally separated (scales)
- We do not need Collins-Soper evolution equation to resum logs of Q^2.
- The resummed logs are the following:



$\ln\frac{q_T}{Q}$ Running of H and Q^2-factor

$\ln\frac{\Lambda_{QCD}}{q_T}$ DGLAP evolution

Conclusions and Outlook

• We have derived the factorization theorem for DY at low q_T.

• We have defined the TMDPDF on-the-LC with the inclusion of the soft function (square root) in it.

• Its properties: free from rapidity divergencies, we can recover the integrated PDF, universal, gauge invariant.

• Its evolution is done only in terms of the hard probe Q.

• We know its AD at 3-loops based on the factorization theorem.

• Going off-the-LC has no special meaning, it is just a regulator (not Collins' way).

• We do not need Collins-Soper evolution equation (there are no rapidity logs). Instead we use the "linearity" argument.

• Generalization to other TMD quantities is straightforward: spin-dependent quantities (Sivers,...), gluon TMDPDF, etc

Back up slides

Matching with QCD

$$M^v_{QCD} = \frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta(1-z)\delta^{(2)}(\vec{q}_\perp)\left[\boxed{-2\ln^2\frac{\Delta}{Q^2} - 3\ln\frac{\Delta}{Q^2}} - \frac{9}{2} + \frac{\pi^2}{2}\right]$$

$$M^v_{SCET} = H(Q^2/\mu^2)\frac{\alpha_s C_F}{2\pi}\delta(1-x)\delta(1-z)\delta^{(2)}(\vec{q}_\perp)\left[\boxed{\frac{2}{\varepsilon^2_{\rm UV}} + \frac{1}{\varepsilon_{\rm UV}}\left(3 + 2\ln\frac{\mu^2}{Q^2}\right)}\right.$$
$$\left.\boxed{-2\ln^2\frac{\Delta}{Q^2} - 3\ln\frac{\Delta}{Q^2}} + 3\ln\frac{\mu^2}{Q^2} + \ln^2\frac{\mu^2}{Q^2} + \frac{7}{2} - \frac{2\pi^2}{3}\right]$$

$$\boxed{H(Q^2/\mu^2) = 1 + \frac{\alpha_s C_F}{2\pi}\left[-3\ln\frac{\mu^2}{Q^2} - \ln^2\frac{\mu^2}{Q^2} - 8 + \frac{7\pi^2}{6}\right]}$$

$$M^r_{SCET} = \frac{\alpha_s C_F}{2\pi^2}\left[\delta(1-x)(1-z)\frac{1}{q_T^2} + \delta(1-z)(1-x)\frac{1}{q_T^2} + \delta(1-x)\frac{2z}{(1-z)_+}\frac{1}{q_T^2}\right.$$
$$\left. + \delta(1-z)\frac{2x}{(1-x)_+}\frac{1}{q_T^2} + 2\delta(1-x)\delta(1-z)\frac{1}{q_T^2}\ln\frac{Q^2}{q_T^2}\right] = M^r_{QCD}$$

Collins' TMDPDF (I)

• The collinear part is kept *on-the-LC*, while the soft subtraction has a complicated structure with 3 soft functions: one *on-the-LC* and 2 *off-the-LC*.

$$\tilde{F}_{f/P}(x,b;\zeta_A,\mu) = \tilde{F}^{\text{unsub}}_{f/P}(x,b;\mu)\sqrt{\frac{\tilde{S}(b;+\infty,y_n)}{\tilde{S}(b;+\infty,-\infty)\,\tilde{S}(b;y_n,-\infty)}}$$

• It can be rewritten as:

$$\tilde{F}_{f/P}(x,b;\zeta_A,\mu) = \frac{\tilde{F}^{\text{unsub}}_{f/P}(x,b;\mu)}{\sqrt{\tilde{S}(b;+\infty,-\infty)}}\sqrt{\frac{\tilde{S}(b;+\infty,y_n)}{\tilde{S}(b;y_n,-\infty)}}$$

$$\zeta_{F,n} = (p^+)^2 e^{-2y_n}$$
$$\zeta_{F,\bar{n}} = (\bar{p}^-)^2 e^{2y_n}$$

This is our TMDPDF

Depends on rapidity cutoff
Gives the Collins-Soper evolution

• Collins' TMDPDF has un-cancelled rapidity divergencies that come form the 2nd factor
• He uses them to resum the logs of Q^2 by Collins-Soper evolution equation.
• The ansatz for zetas has no meaning for individual TMDPDF (far off-the-LC!!). But when combined both, they cancel and get the Q^2 of the hadronic tensor. This facto motivates the choice for zetas.

Collins' TMDPDF (II)

• Collins' way of going off.the-LC is "strange".
• The parameter that parametrizes the "off-the-light-cones" should take two different limits at the same time in order to come back to the light-cone.
• Once he combines two resummed TMDPDFs into the hadronic tensor, the zetas cancel. Then one thinks to make an ansatz over their value for each individual TMDPDF.

$$n_t = (1, -e^{-2y_n}, 0_\perp)$$
$$\bar{n}_t = (-e^{2y_n}, 1, 0_\perp)$$

Two different limits for y_n to recover the LC!!

• Collins takes a counter-term for the TMDPDF that includes rapidity divergencies, and this is why his AD depends on y_n:

$$\gamma_{n1}^{Collins} = \frac{\alpha_s C_F}{4\pi}\left[6 + 4\ln\frac{\mu^2}{Q^2 e^{-2y_n}}\right]$$

• If y_n is taken to be 0, then we are **far off-the-light-cone!!**
• The way of going off-the-light-cone of, for instance [Ji, Ma, Yuan '03], is different: just one limit of some parameter is necessary to come back to the light-cone.

Collins' TMDPDF vs Ours

$$\tilde{F}_{f/P}(x,b;\zeta_{F,f},\mu_f) = \overbrace{\sum_{j=q,g}\int_x^1 \frac{dx'}{x'}\tilde{C}_{f/j}\left(\frac{x}{x'},b^*;\zeta_{F,i}=\mu_{b^*}^2,\mu_i=\mu_{b^*}\right) f_{j/P}(x';\mu_{b^*})}^{\mathrm{A}}$$

$$\times \overbrace{\exp\left\{\tilde{K}(b^*;\mu_{b^*})\ln\frac{\sqrt{\zeta_{F,f}}}{\mu_{b^*}} + \int_{\mu_i=\mu_{b^*}}^{\mu_f}\frac{d\mu'}{\mu'}\left[\gamma_F\left(\alpha_s(\mu');\ln\frac{\zeta_{F,f}}{\mu'^2}\right)\right]\right\}}^{\mathrm{B}} \times \overbrace{\tilde{F}^{NP}_{f/P}(x,b;\zeta_{F,f},\zeta_{F,0})}^{\mathrm{C}}$$

- Ours is much more simple than Collins' formalism.
- Compatible with factorization theorem: we can resum logs for DY without Collins-Soper.
- We can go to higher orders in perturbation theory.
- Out TMDPDF can be calculated with any regulator, even going off-the-light-cone (ala Ji, Ma, Yuan '05, for instance)