Particle Accelerators and Beam Optics

Mike Syphers

Northern Illinois University

Fermi National Accelerator Laboratory

Exotic Beams Summer School
Argonne National Laboratory

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msyphers@niu.ed

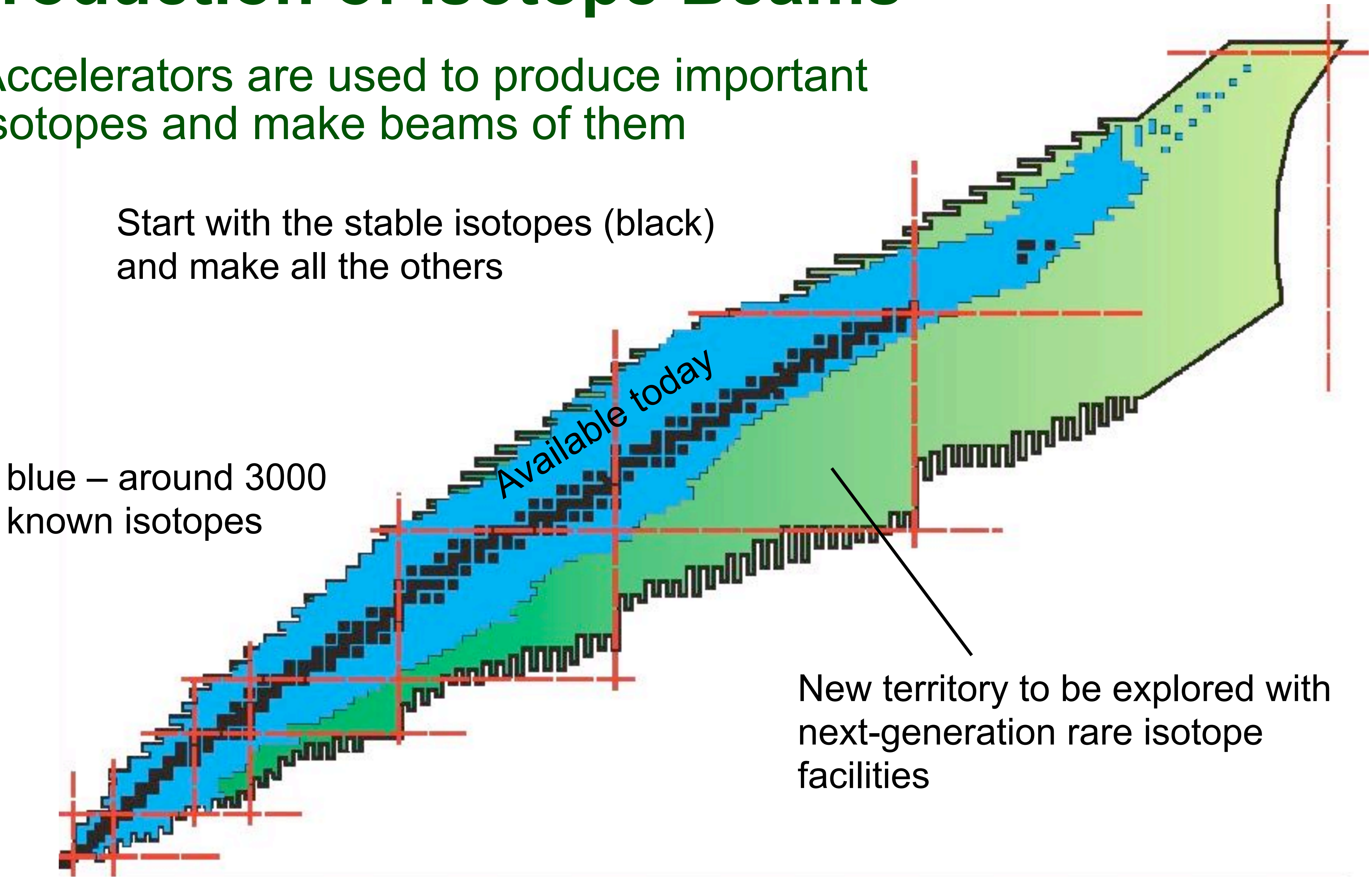
Outline

- Lecture 1
  - Overview of types and uses of accelerators
  - Single-pass vs. repetitive systems
  - Transverse vs. longitudinal motion
  - Beams and particle distributions
  - Transverse beam optics

- Lecture 2
  - Dispersion
  - Longitudinal beam dynamics
    - buncher, re-bunchers; buckets and bunches
  - Optics modules
  - Accelerators for nuclear physics
  - Light sources
  - Accelerators for high energy physics
  - Future directions

Production of Isotope Beams

- Accelerators are used to produce important isotopes and make beams of them



Production of Rare Isotope Beams

- There is a variety of nuclear reaction mechanisms used to add or remove nucleons:
  - Spallation, Fragmentation, Coulomb fission (photo fission), Nuclear induced fission, Light ion transfer, Fusion-evaporation (cold, hot, incomplete, …), Fusion-Fission, Deep Inelastic Transfer, Charge Exchange, ...
- The accelerator system produces a primary beam of charged particles and delivers them to a target
  - e.g., use protons for spallation, heavy ions for fragmentation, *etc.*
- *Thus,* once created in a source, need to accelerate ions, direct them along a desired trajectory, and keep them contained along the way

accelerating devices

steering devices

focusing devices

Accelerators

- Various types...
  - Cyclotron
    - » NSCL, GANIL, TRIUMF (proton driver), HRIBF (proton driver), RIKEN RIBF
  - Synchroton
    - » GSI, FAIR-GSI
  - Linear Accelerator (*LINAC*): ATLAS (ANL), FRIB (MSU)
  - Others like Fixed-Field Alternating Gradient accelerators currently not used

- Main Parameters
  - Top Energy (*e.g*. FRIB will have 200 MeV/u uranium ions)
  - Particle range (single particle, or multiple particle species)
  - Intensity or Beam Power
    - » beam intensity: # particles /sec = $dN/dt = I/Qe$ 1 pμA == 6×10^{12} /s
    - » *Power* = *dN/dt* [pμA] x *Particle Energy* [GeV] *e.g*., 400 kW = 8 [pμA] x 50 [GeV]
    - » If particle of mass Am_u and charge Qe has energy E, then can write
      - *Power* [W] = $dN/dt \times E = dN/dt \times (A \times E/A)$ == $(A/Q) \times I$ [μA] x E/A [MeV/u]/e
      - etc., ...

A Little Accelerator History

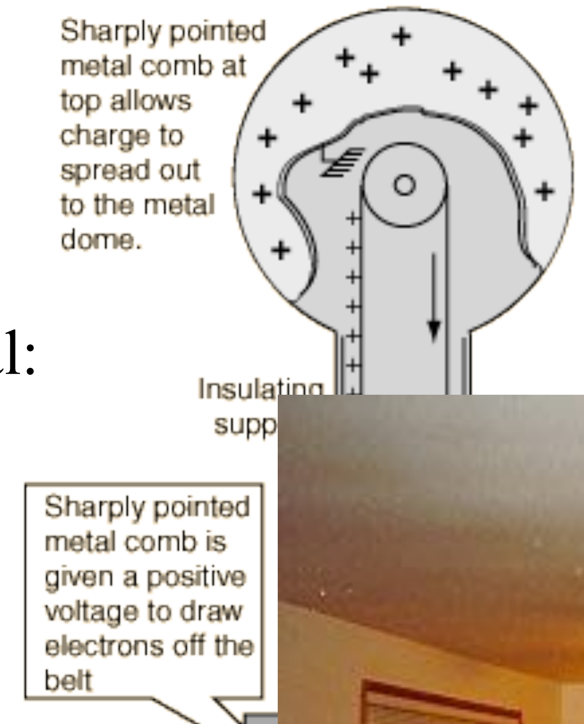
- DC Acceleration

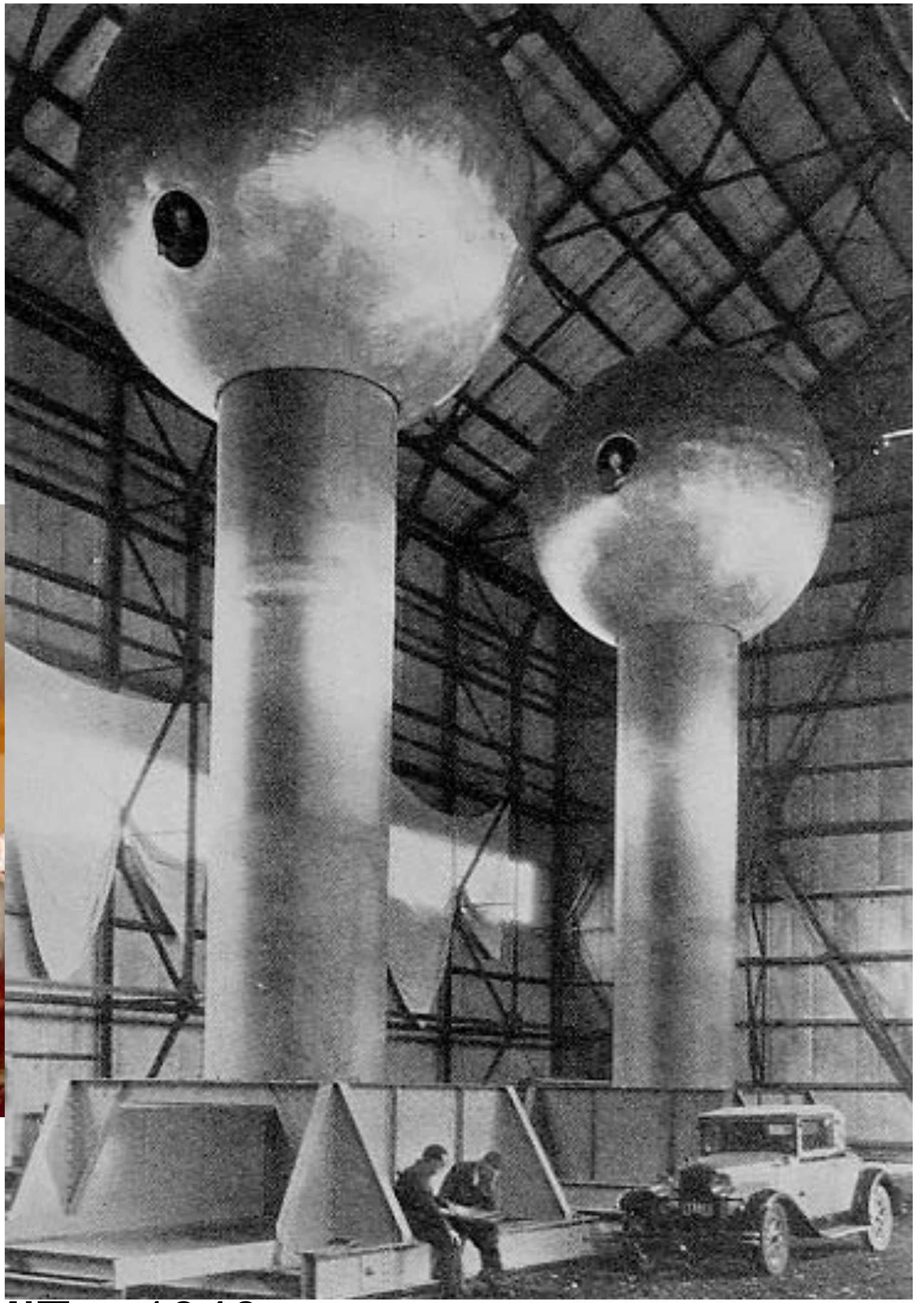
1927: Lord Rutherford requested a "copious supply" of projectiles more energetic than natural alpha and beta particles. At the opening of the resulting High Tension Laboratory, Rutherford went on to reiterate the goal:

"What we require is an apparatus to give us a potential of the order of 10 million volts which can be safely accommodated in a reasonably sized room and operated by a few kilowatts of power. We require too an exhausted tube capable of withstanding this voltage… I see no reason why such a requirement cannot be made practical."

Van de Graaff

(1929)





MIT, c.1940s

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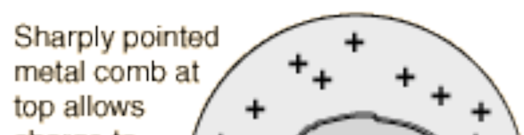
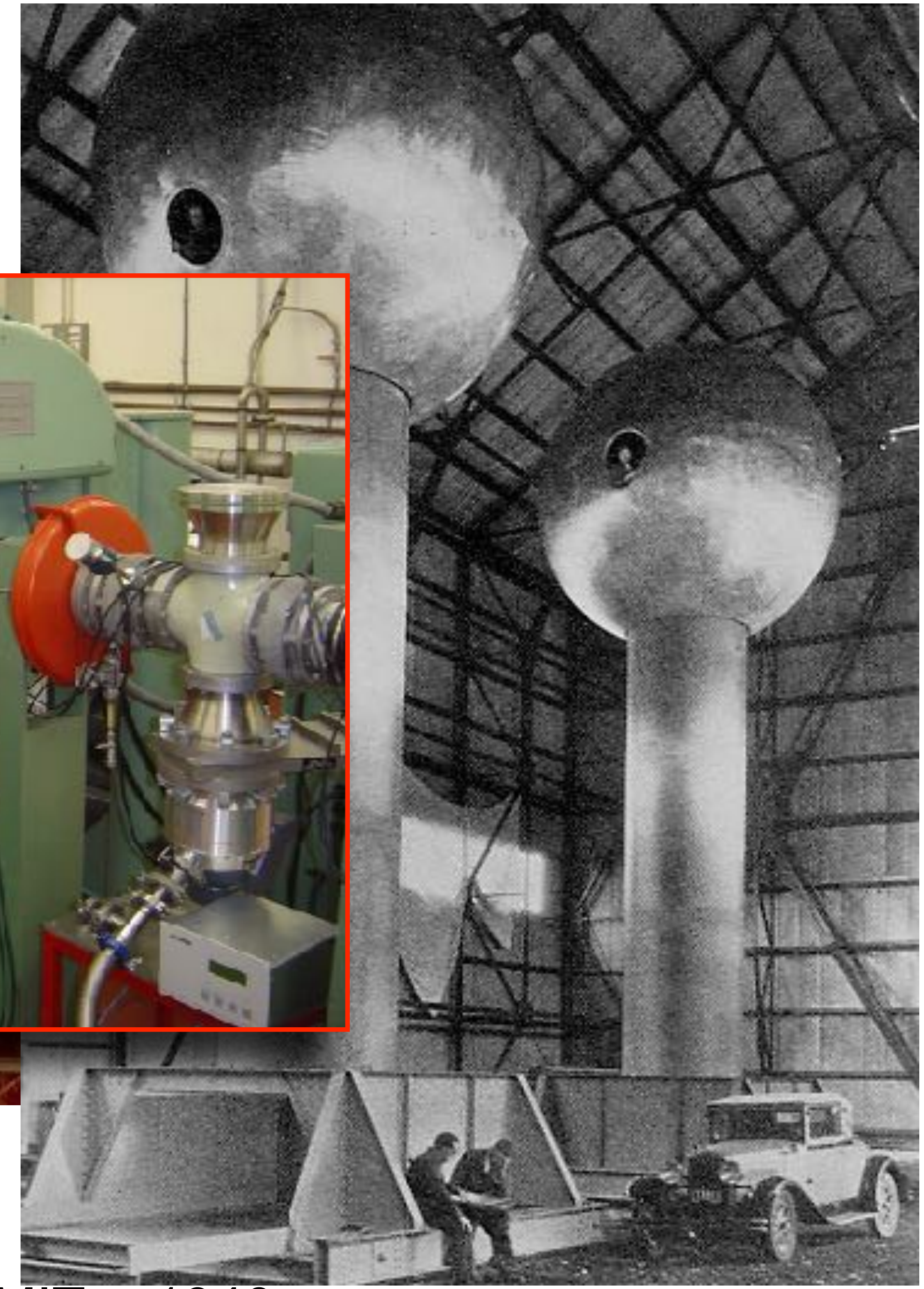




MIT, c.1940s

Cockcroft and Walton

- Voltage Multiplier

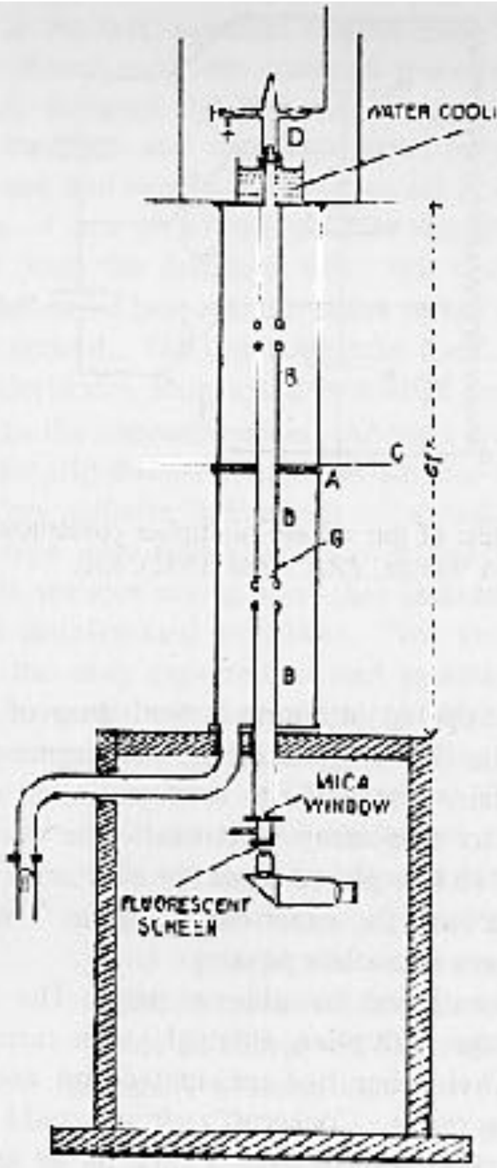
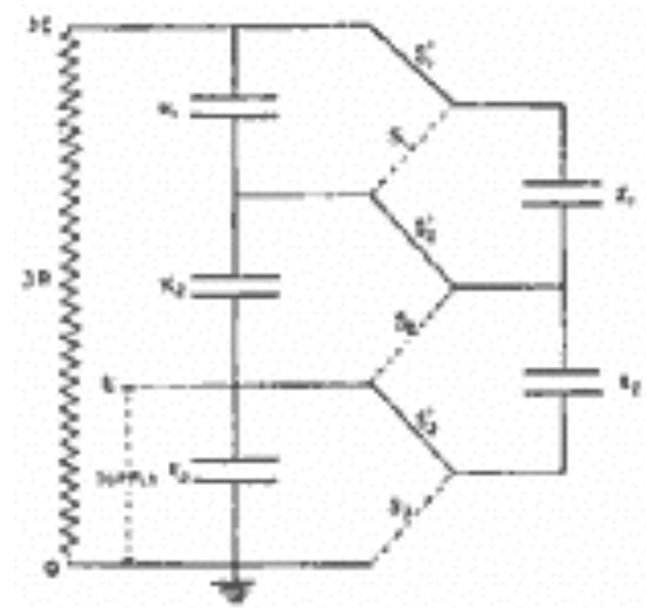
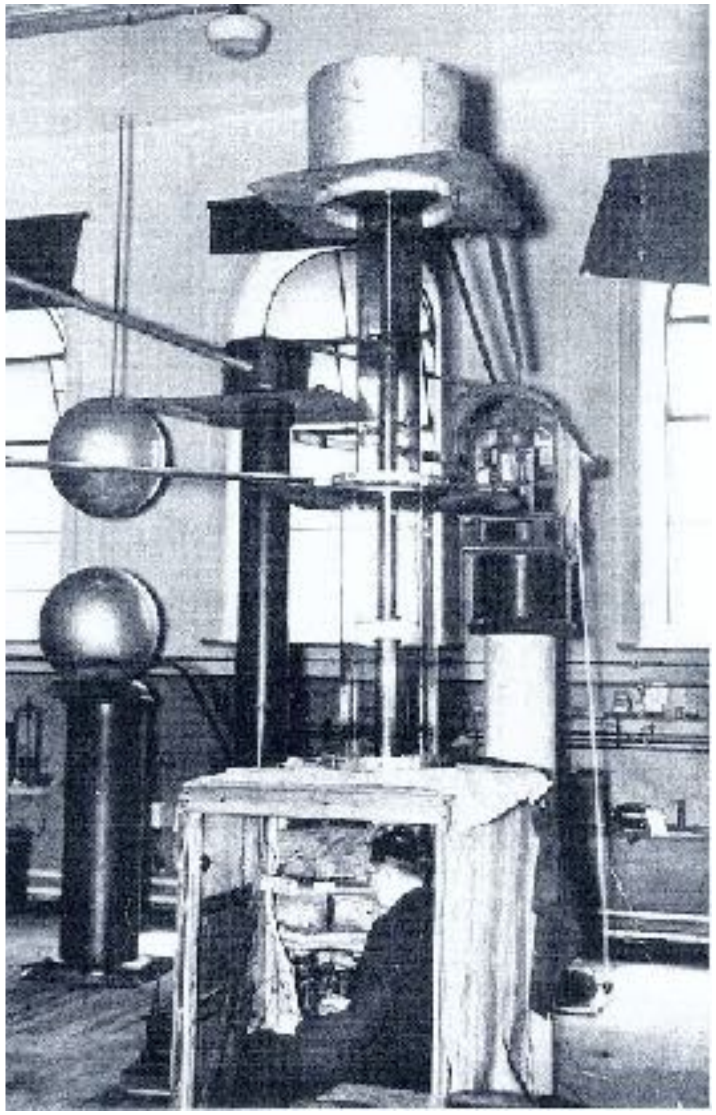


FIG. 2.11 Accelerating tube and target arrangement of the Cockcroft-Walton machine. The source is at D, C is a metallic ring joint between the two sections of the constantly pumped tube. The mica window closes the evacuated space. Cockcroft and Walton, *PRS, A136* (1932), 626.



Converts AC voltage V to DC voltage n x V



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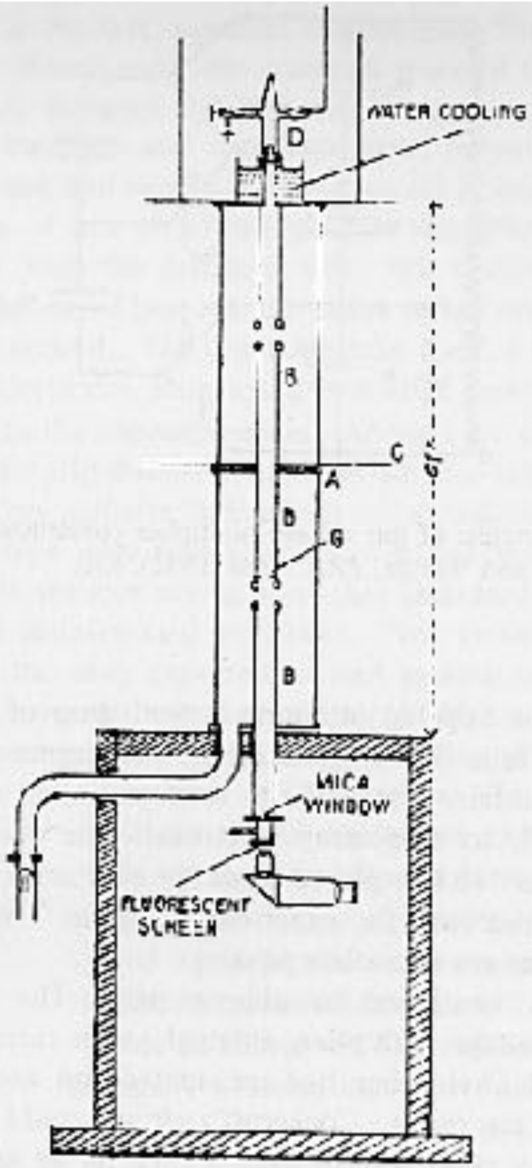

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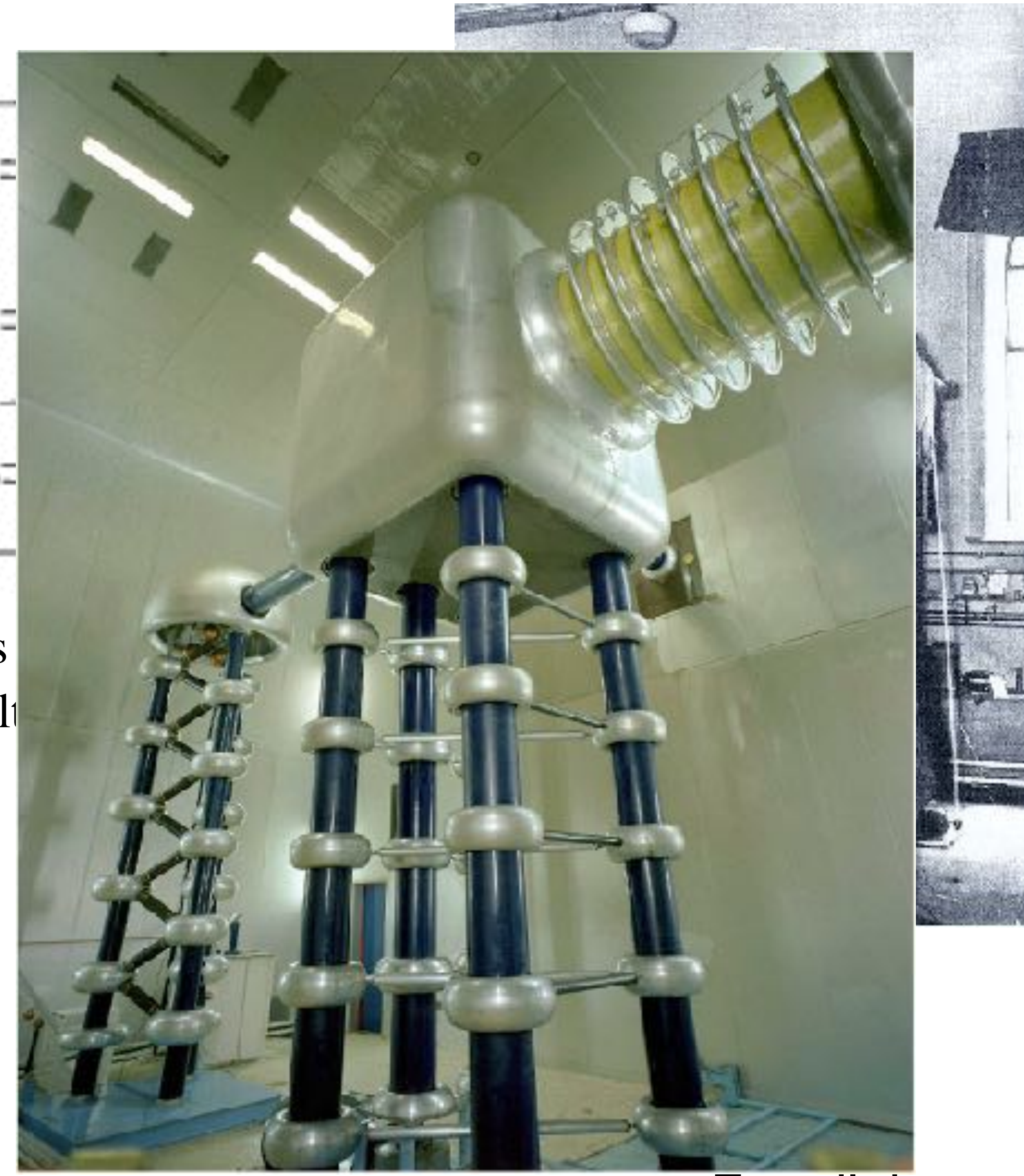
Converts
DC volt

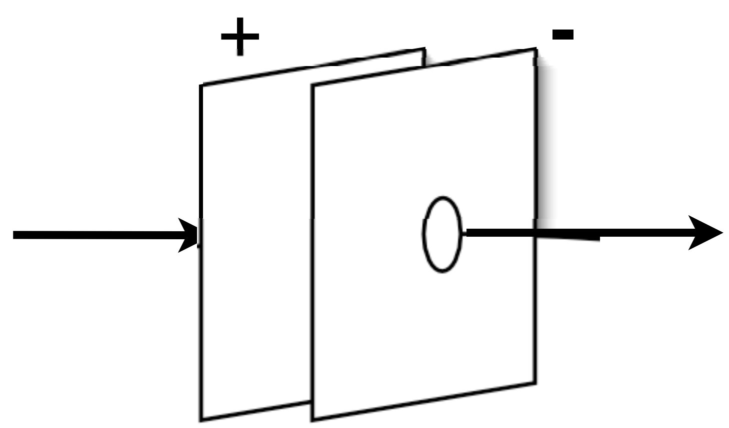


Fermilab

The Route to Higher Energies

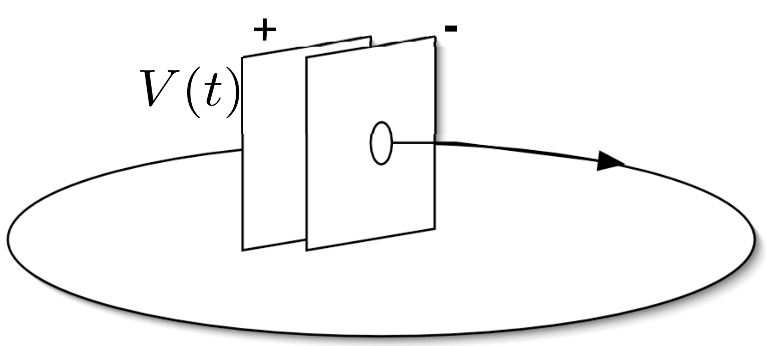
- The need for AC systems

$$\text{energy gain} = q \cdot V$$

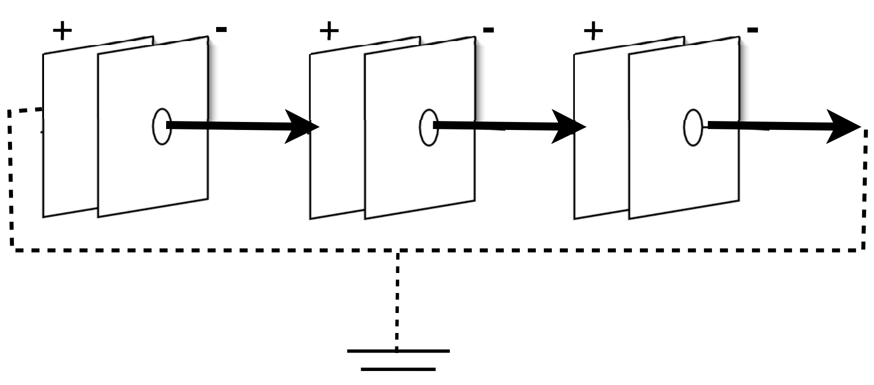


DC systems limited to a few MV

Circular Accelerator



Linear Accelerator



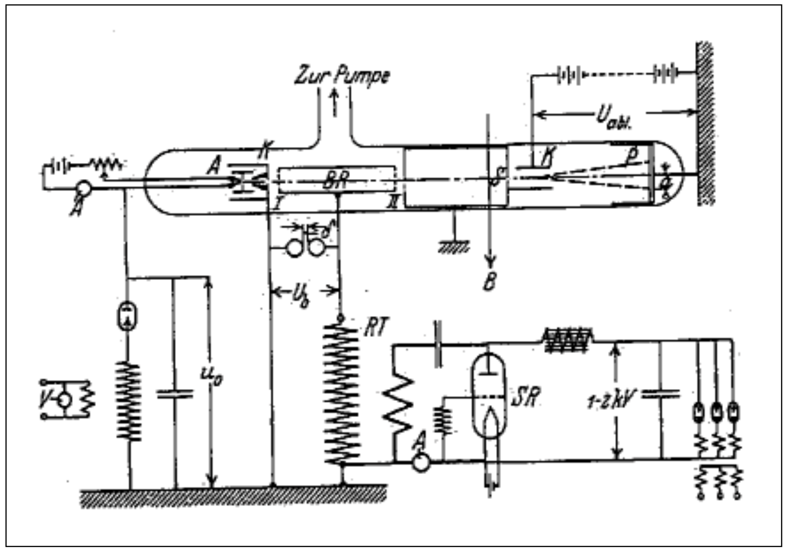
$$\oint (q\vec{E}) \cdot d\vec{s} = work = \Delta(energy)$$

To gain energy, a time-varying field is required:

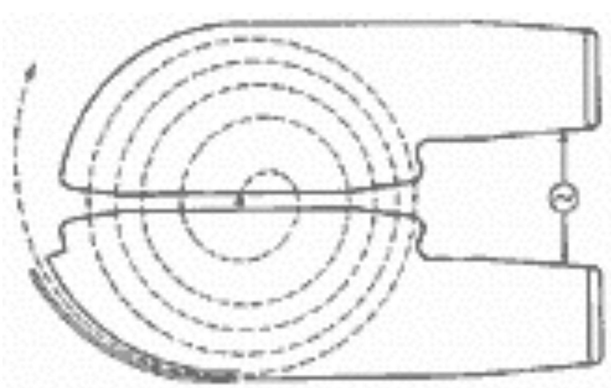
$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$

Oscillating Fields

- The linear accelerator (linac) -- 1928-29
  - Wideroe (U. Aachen; grad student!)
    - Dreamt up concept of "Ray Transformer" (later, called the "Betatron"); thesis advisor said was "sure to fail," and was rejected as a PhD project. Not deterred, illustrated the principle with a "linear" device, which he made to work -- got his PhD in engineering
  - 50 keV; accelerated heavy ions (K+, Na+)
  - utilized oscillating voltage of 25 kV @ 1 MHz



- The Cyclotron -- 1930's, Lawrence (U. California)
  - read Wideroe's paper (actually, looked at the pictures!)
  - an extended "linac" unappealing -- make it more compact:

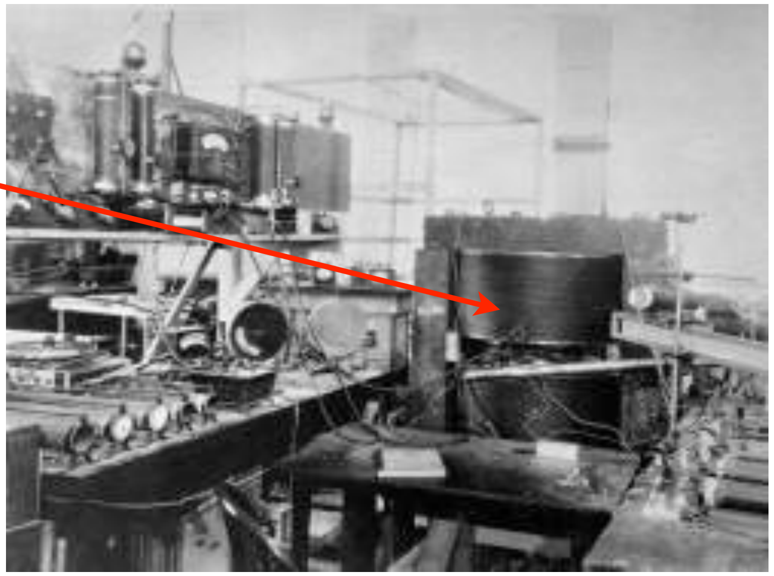


V

$$\frac{1}{T} = \frac{q \cdot B}{2\pi m}$$

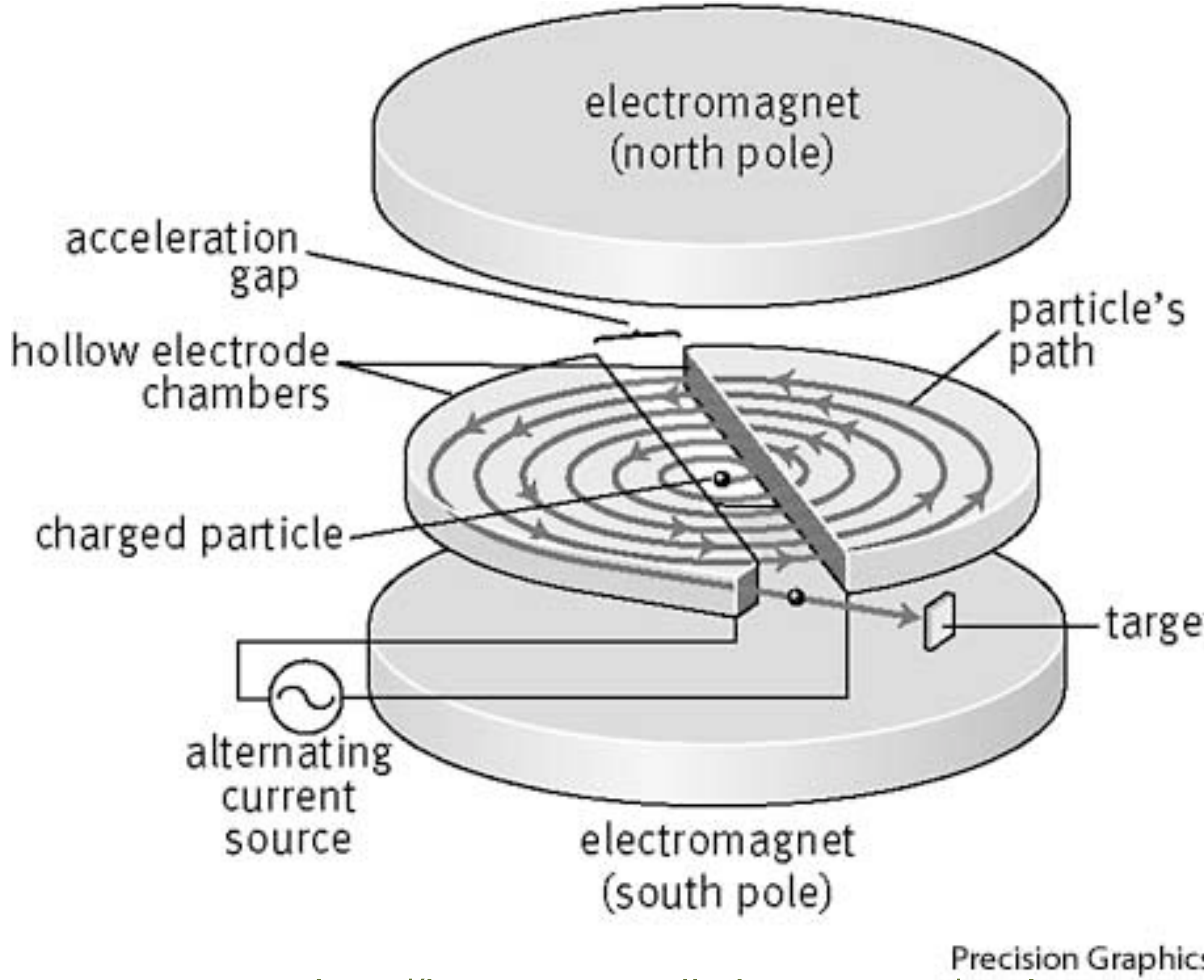
4.5 inch diameter!

11 inch diameter



Cyclotrons

- Relatively easy to operate and tune (only a few parts).
- Tend to be used for isotope production and places where reliable and reproducible operation are important
- Intensity is moderately high, acceleration efficiency is high, cost low
- Relativity is an issue, so energy is limited to a few hundred MeV/u.
- RIKEN Superconducting Ring Cyclotron 350 MeV/u



http://images.yourdictionary.com/cyclotron

60-inch Cyclotron, Berkeley -- 1930's

184-inch Cyclotron, Berkeley -- 1940's

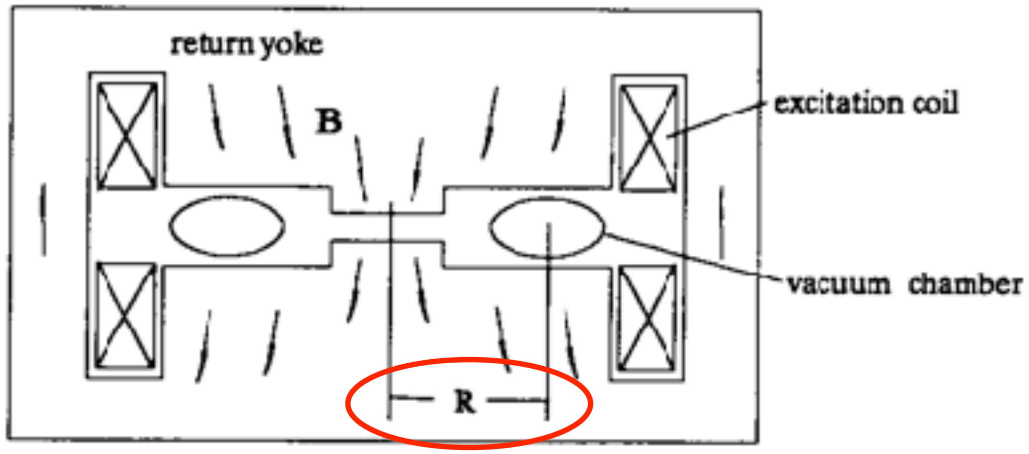
Meeting up with Relativity

- The Synchrocyclotron (FM cyclotron) -- 1940's
  - beams became relativistic (esp. e^-) --> oscillation frequency no longer independent of momentum; cyclotron condition no longer held throughout process; thus, modulate freq.

- The Betatron -- 1940, Kerst (U. Illinois)
  - induction accelerator

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$

    - »used for electrons
    - »Beam dynamics heavily studied
      - "betatron oscillations"



field on orbit of radius *R*

$$\boxed{\frac{d\Phi}{dt} = 2(\pi R^2)\frac{dB_z}{dt}}$$

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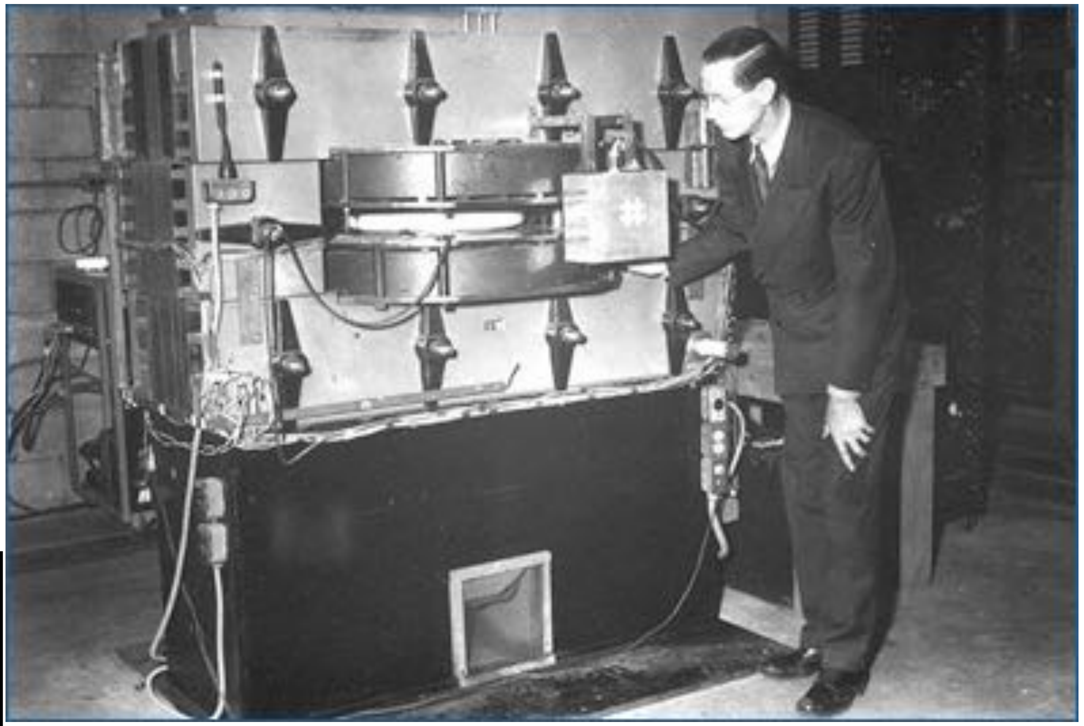
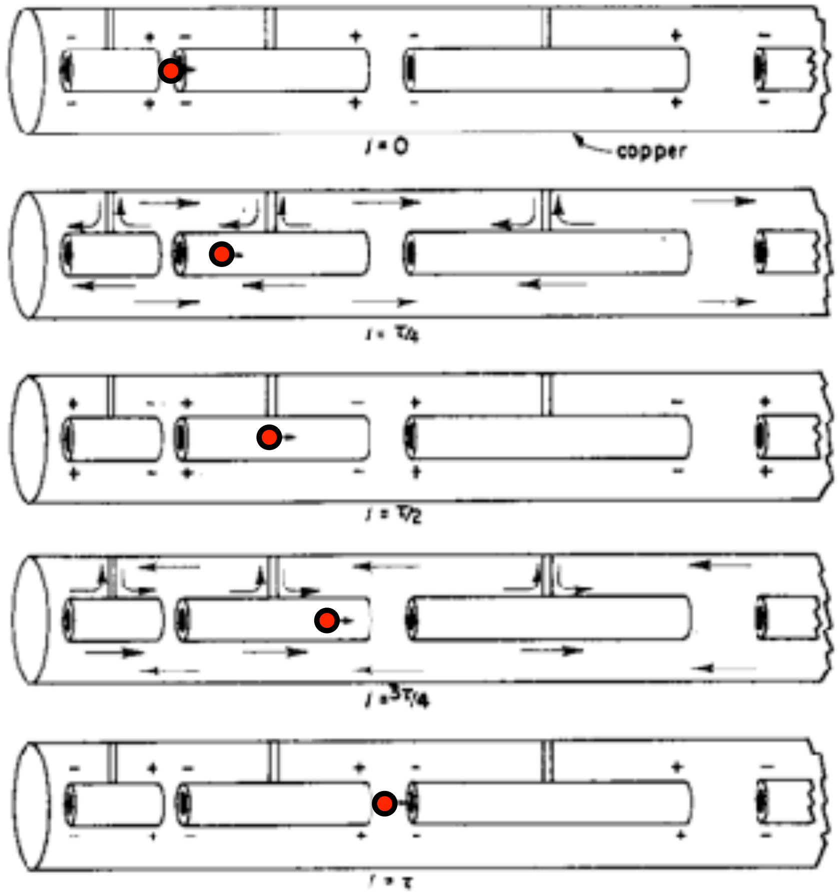


~ 2 MeV; later models --> 300 MeV

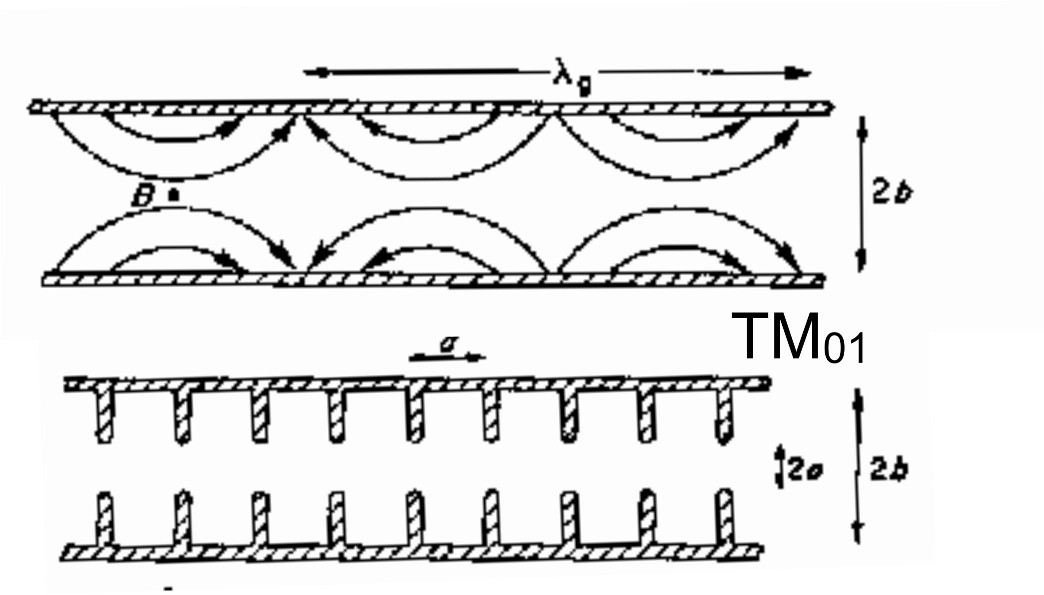
- The Microtron --1944, Veksler (Russia)
  - use one cavity with one frequency, but vary path length each "revolution" as function of particle speed

The "Modern" Linear Accelerator

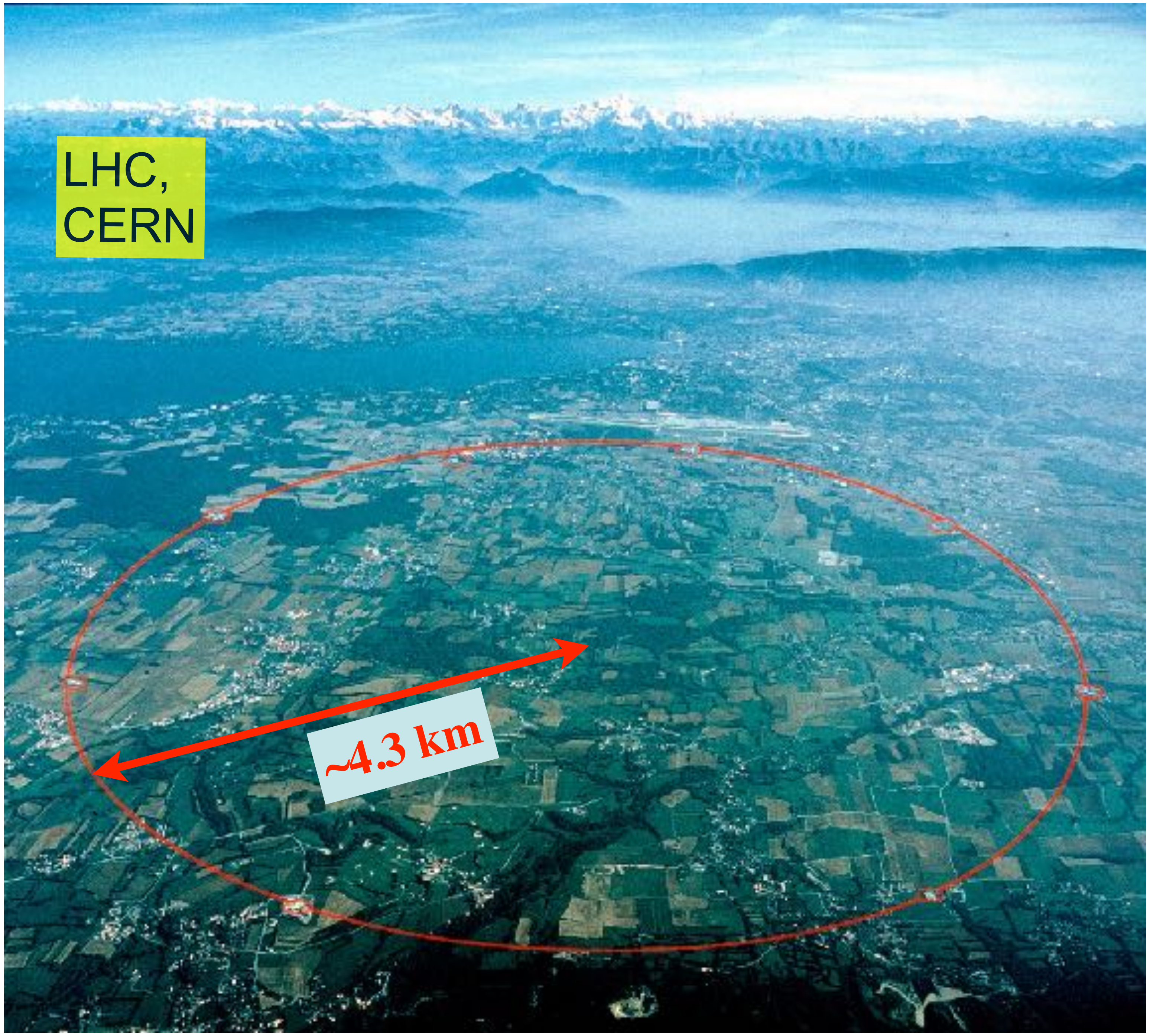


- Alvarez -- 1946 (U. California)
  - cylindrical cavity with drift tubes
  - particles "shielded" as fields change sign
  - most practical for protons, ions
  - GI surplus equip. from WWII Radar technology

- Traveling-Wave Electron Accelerator -- c.1950 (Stanford, + Europe)
  - TM_{01} waveguide arrangement
  - iris-loaded cylindrical waveguide
    - »match phase velocity w/ particle velocity...

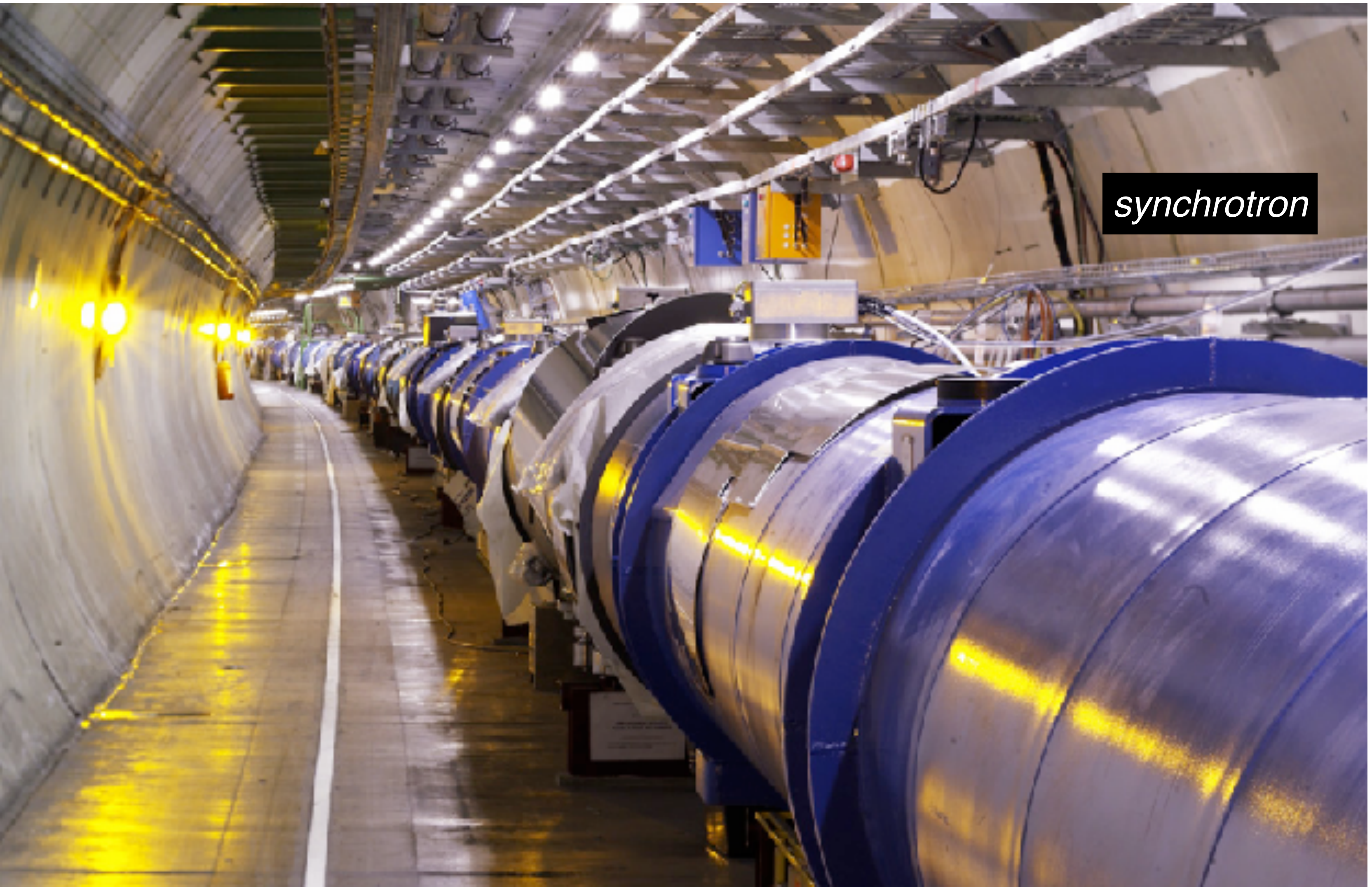


The Large Collider Synchrotrons



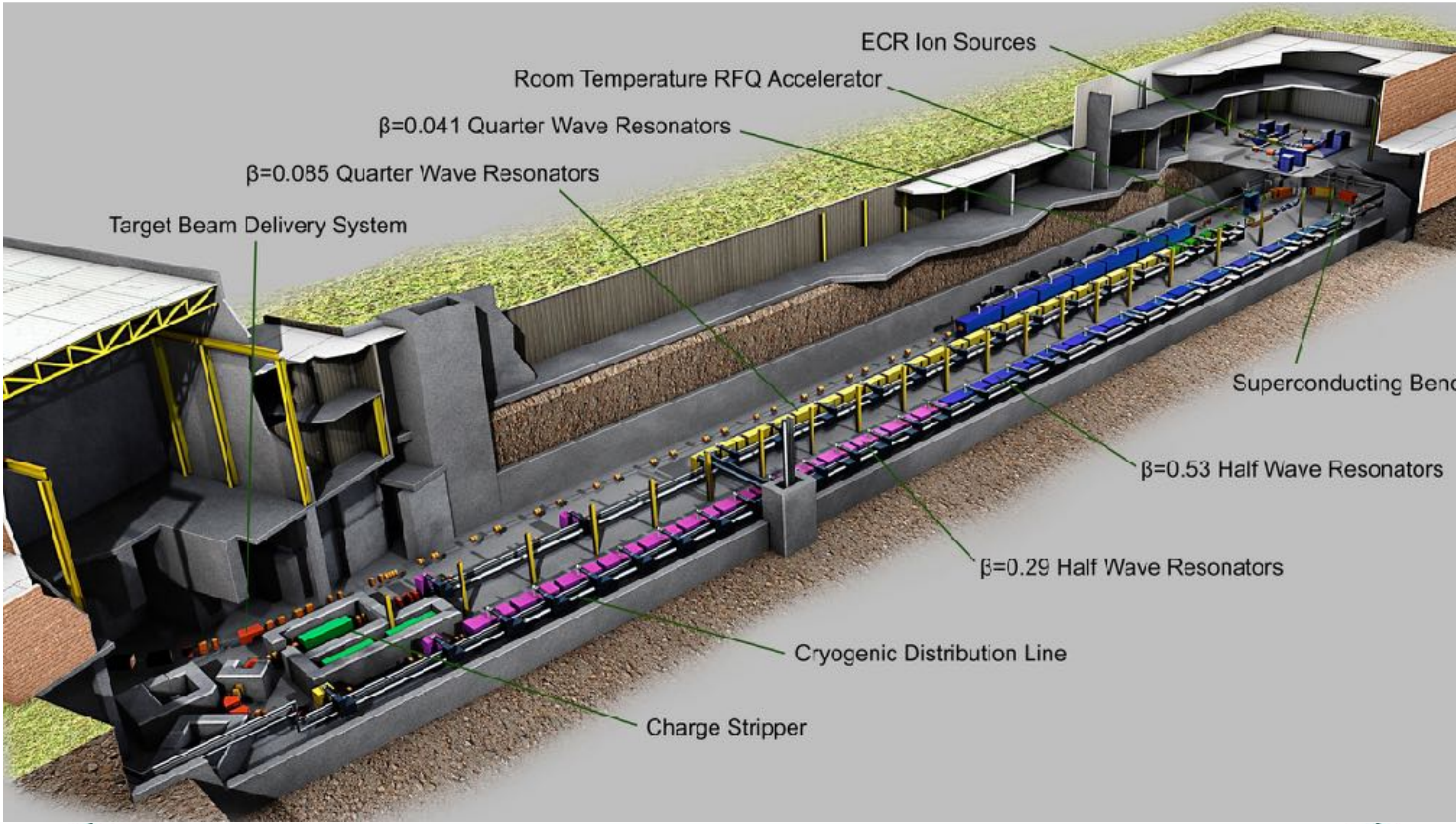
Recent Large-Scale Accelerators

Large Hadron Collider (LHC)



Spallation Neutron Source (SNS)

MSU's Facility for Rare Isotope Beams (FRIB)



Future High Energy Facilities

- Groups around the world are looking into the next steps toward even larger accelerators for fundamental physics research

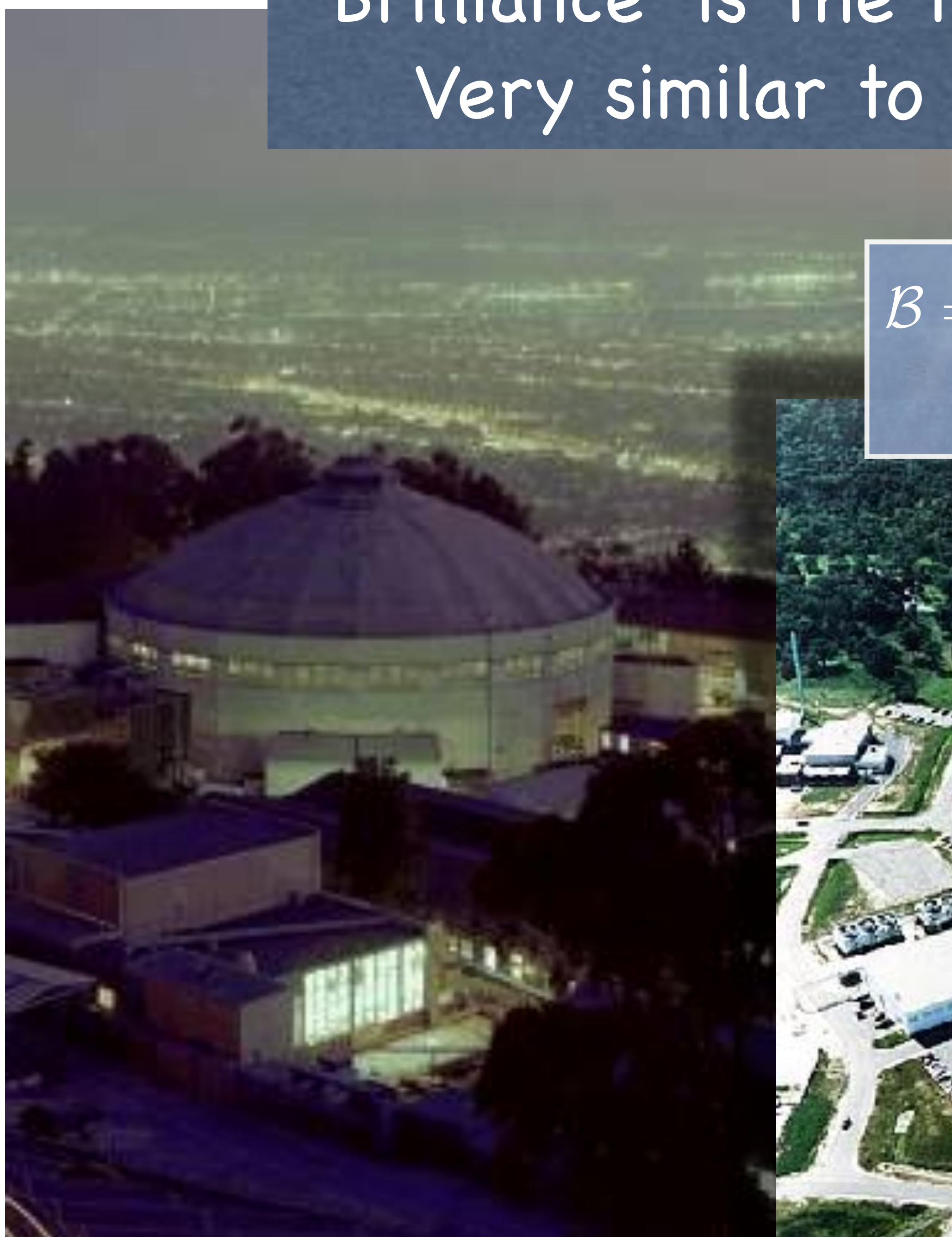
- Next-generation Hadron collider
- Next-generation Lepton collider

view from France into Switzerland, showing existing LHC complex (orange) and a possible 100 TeV collider ring (yellow)
photo courtesy J. Wenninger (CERN)

Light Sources

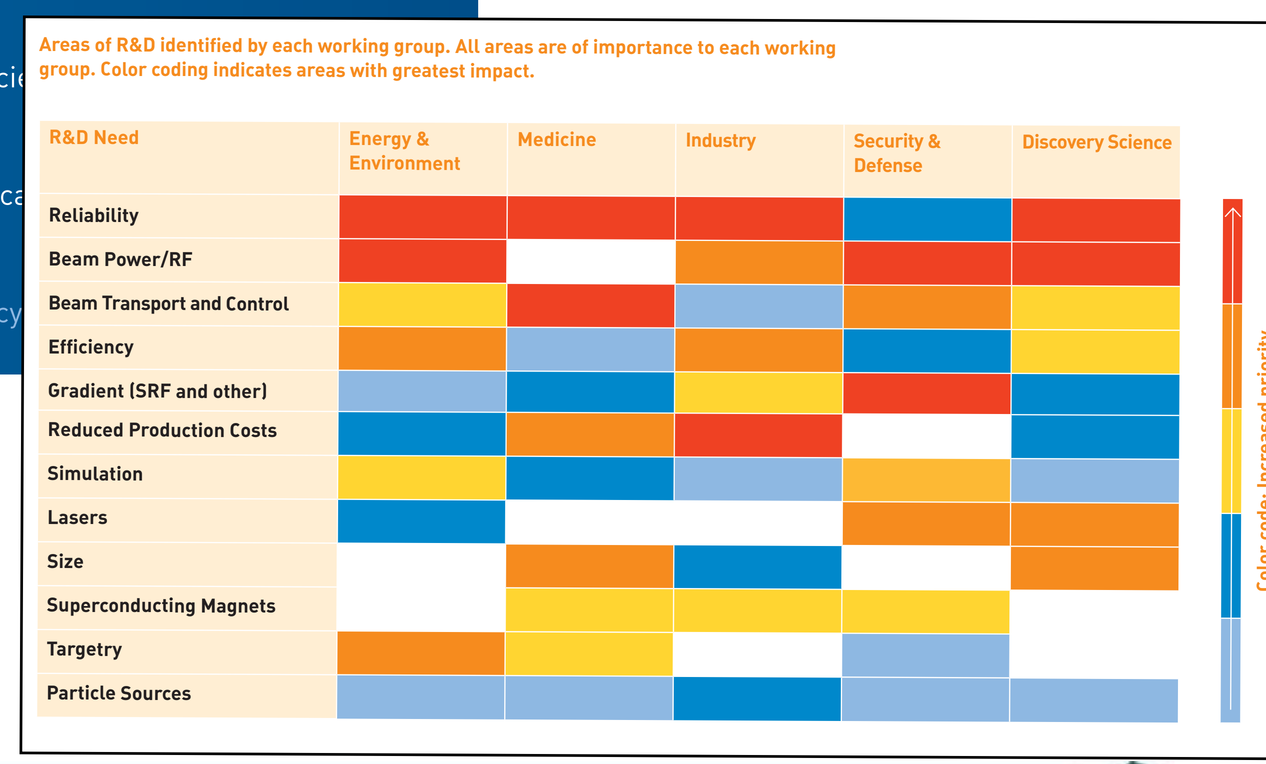
"Brilliance" is the figure of merit
Very similar to luminosity:

$$\mathcal{B} = \frac{\text{photons/sec}}{\text{mm}^2\text{mrad}^2(0.1\%\ \text{BW})}$$



Accelerators for America's Future

- Symposium and workshop held in Washington, D.C., October 2009
- 100-page Report available at web site



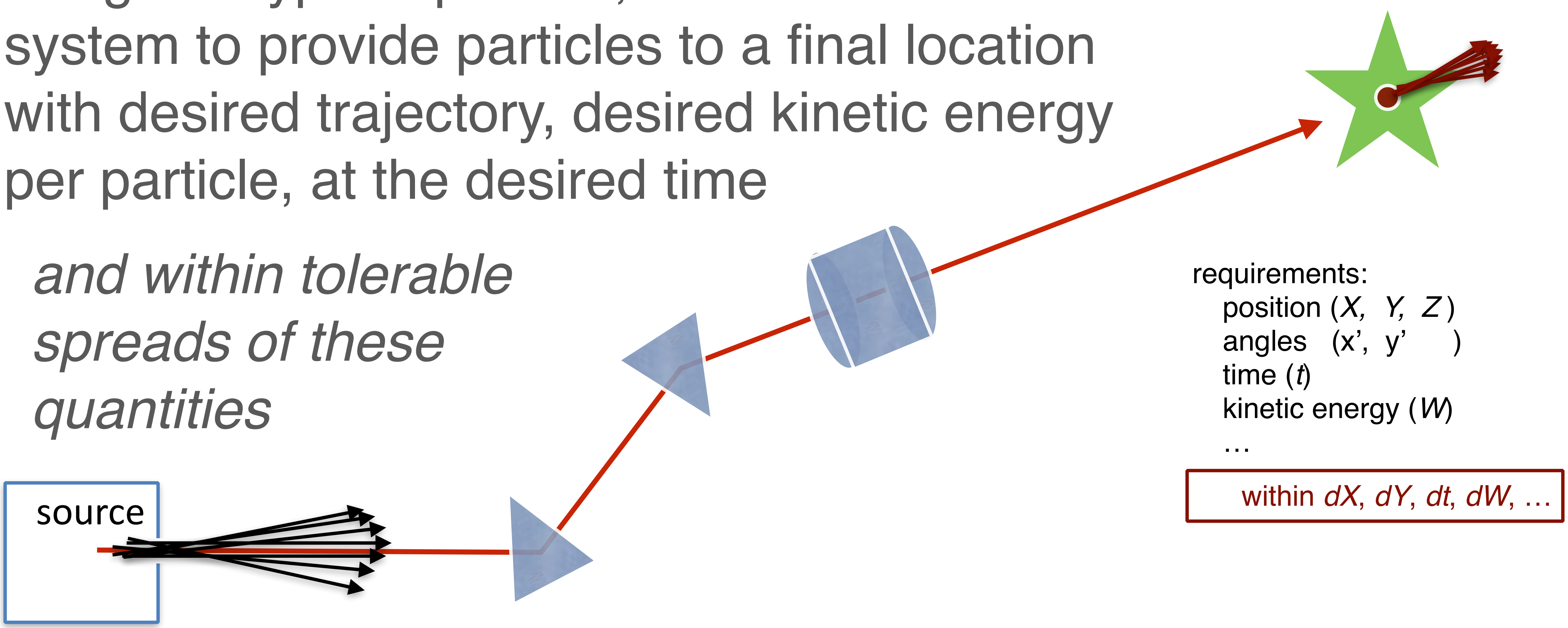
http://www.acceleratorsamerica.org/

The Problem

- 1927: Lord Rutherford requested a "copious supply" of projectiles "more energetic than natural alpha and beta particles"

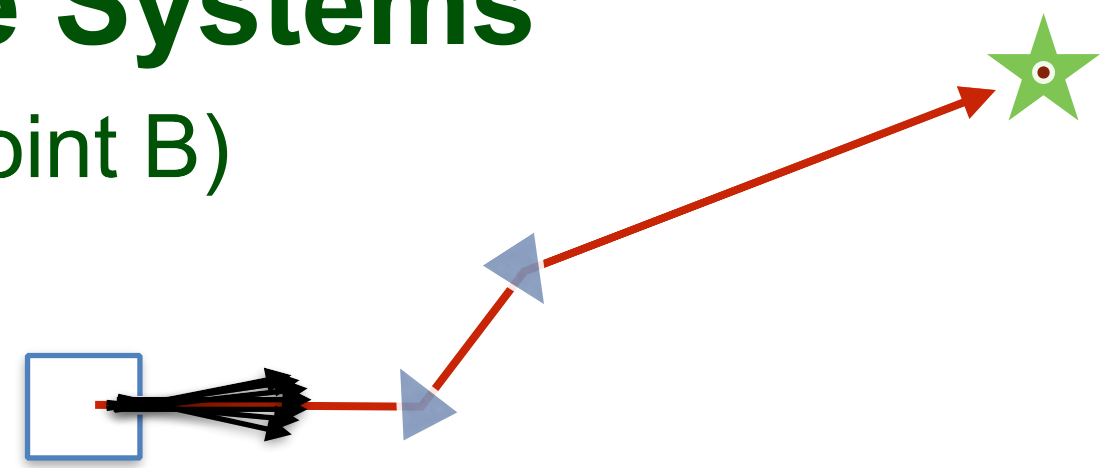
- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time

  and within tolerable spreads of these quantities

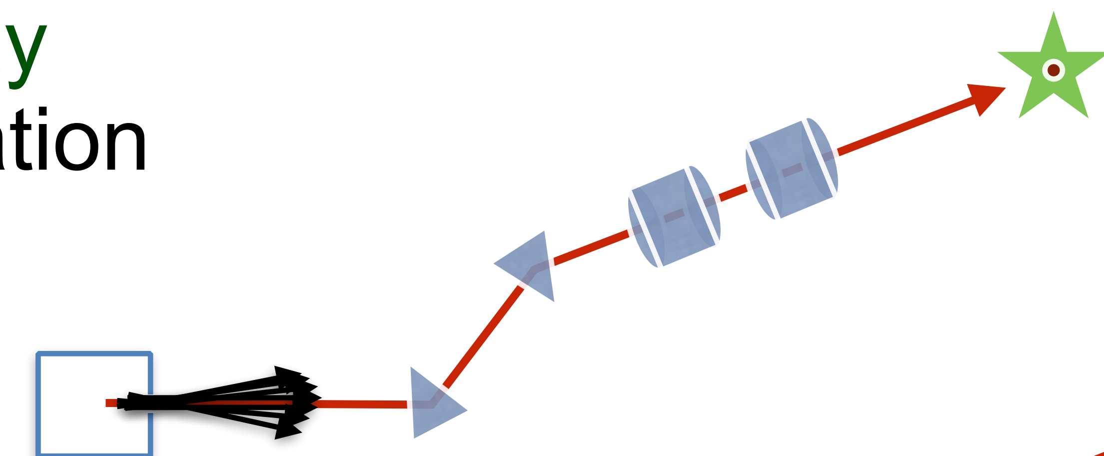


Single-Pass vs. Repetitive Systems

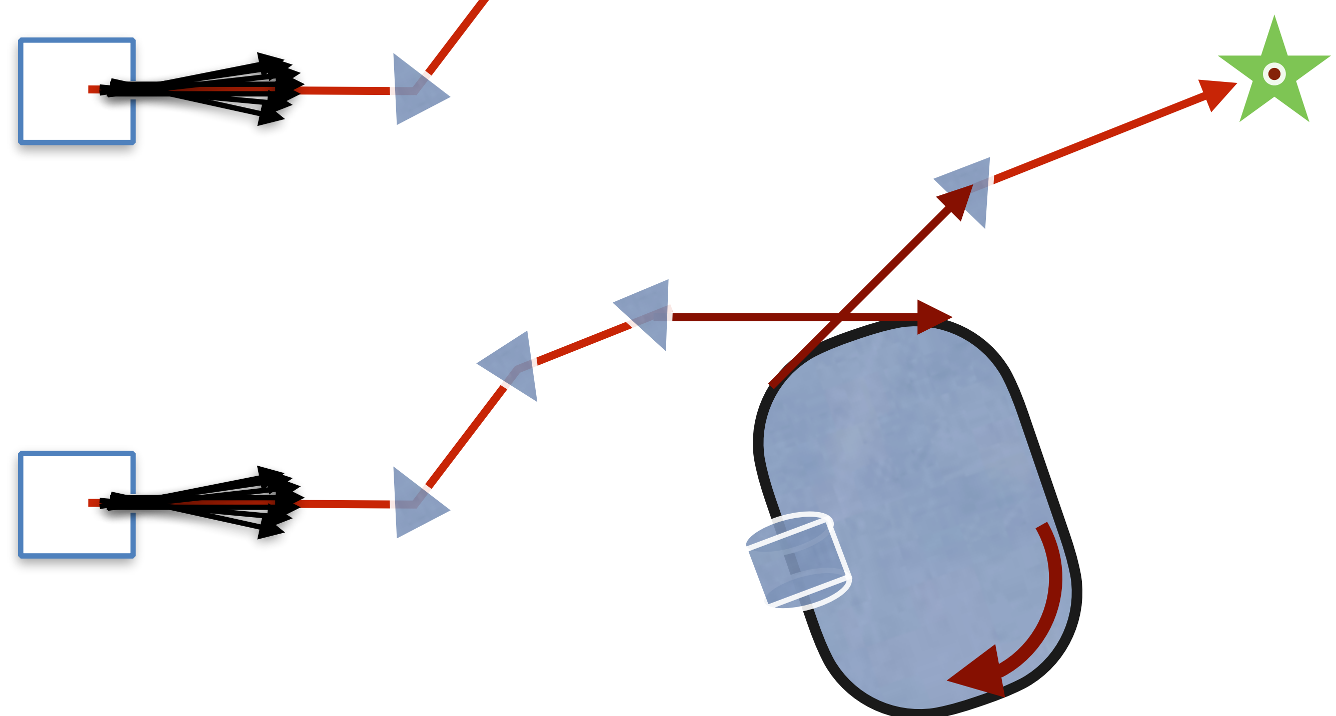
- Beam Transport (from point A to point B)



- Acceleration along the way
  - single-pass with acceleration



  - multi-pass acceleration



may need motion in such a system to be stable for many (millions or more?) revolutions

A Few Words on Particle Sources…

- Electrons — relatively easy
  - filaments; photocathodes, laser driven plasmas,…
- Protons — not “too” hard
  - ionized hydrogen gas, plasma sources,…
- Ions — similar techniques
  - ovens, plasma sources, ECRs — plus, separation
- Even more exotic particles: target, separate, collect
  - heavy ion isotopes
  - pions, muons, antiprotons, neutrinos,…
- Also polarized sources, …

Reduction of the Problem

- Will treat transverse motion of particles through the accelerator as independent of the longitudinal motion, and study these two cases separately. Must show along the way that this is viable approach.
- Certainly not always be the case...
  - electric fields used for focusing at low energies can also accelerate the particles as well;
  - fields in the gaps of cavities will have focusing effects; etc.
- However, much of the "cross talk" can be minimized, and for much of the particle's journey, especially at higher energies, the major transverse focusing can be performed by magnetic fields -- particle's energy not changed
- Look at "linear" fields, *i.e.* linear restoring forces

Stability of Motion Near the Ideal

- Not all particles (any??) begin "on" the design trajectory with *exactly* the ideal energy/momentum
- We wish to have a system that will keep particles near the ideal conditions as they are transported (and possibly accelerated) through the system
- Particles emerge from their "source" with a slight divergence and will need to be guided back toward the ideal trajectory
- Also, particles with different energies/momenta will travel at different speeds, and hence may not arrive at cavities, experiments, etc., at the ideal time

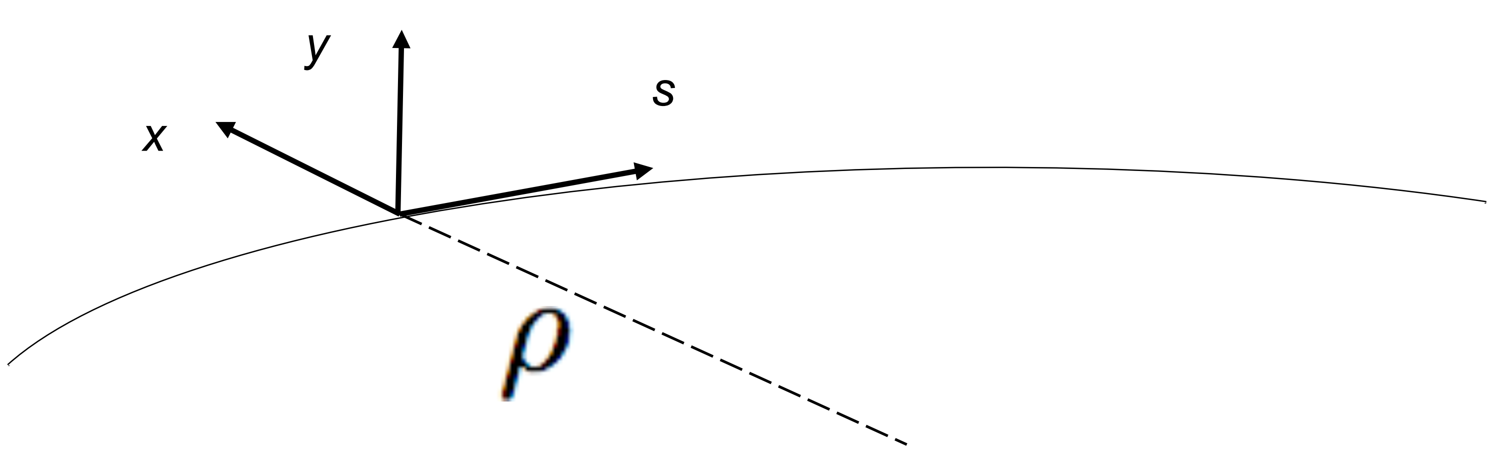
Equations of Motion

- Lorentz Force: $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

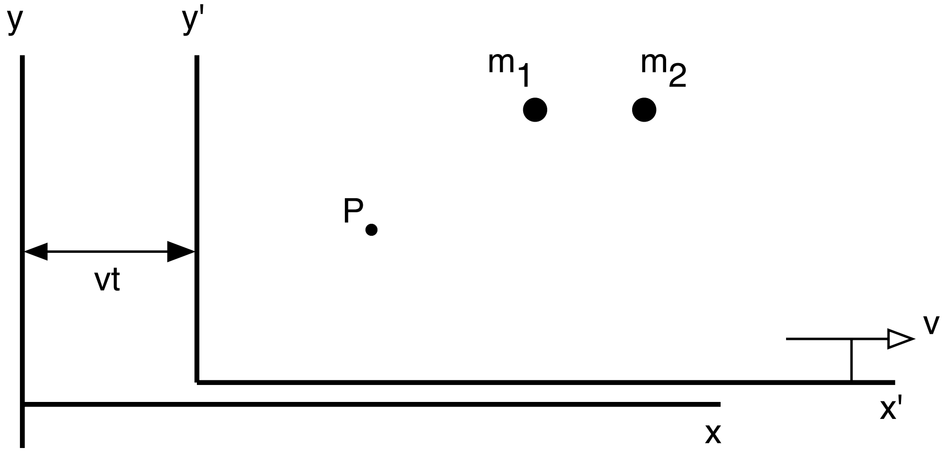
- Magnetic Rigidity
  - particle of unit charge, *q* = *e*: $$B\rho \equiv \frac{p}{q} = \frac{p}{e}$$
  - ion w/ mass *A* (atomic units, u), charge *Q*: $$B\rho = \frac{A}{Q}\left(\frac{1}{300}\frac{\mathrm{T \cdot m}}{\mathrm{MeV/c/u}}\right) p_u$$

- Need for Transverse Focusing
  - Not all particles (any?) begin "on" the design trajectory; need to keep particles nearby

- Reference Trajectory
  - Local Coordinate System



Relativity Reminders…



- Particle energy in the lab

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- total energy: $$E = \gamma mc^2 = mc^2 + W$$
- kinetic energy: $$W = (\gamma - 1)mc^2$$

$$E^2 = (mc^2)^2 + (pc)^2$$

$W \ll mc^2, \ldots$

$$pc = \sqrt{W^2 + 2mc^2 W} \approx \sqrt{2mc^2 W}$$

$$p \approx \sqrt{2mW}$$

Some Examples…

- For $W << mc^2$:

$$p \approx \sqrt{2mW} \qquad pc \approx \beta mc^2 = \sqrt{2mc^2W} \qquad \beta = \sqrt{\frac{2W}{mc^2}}$$

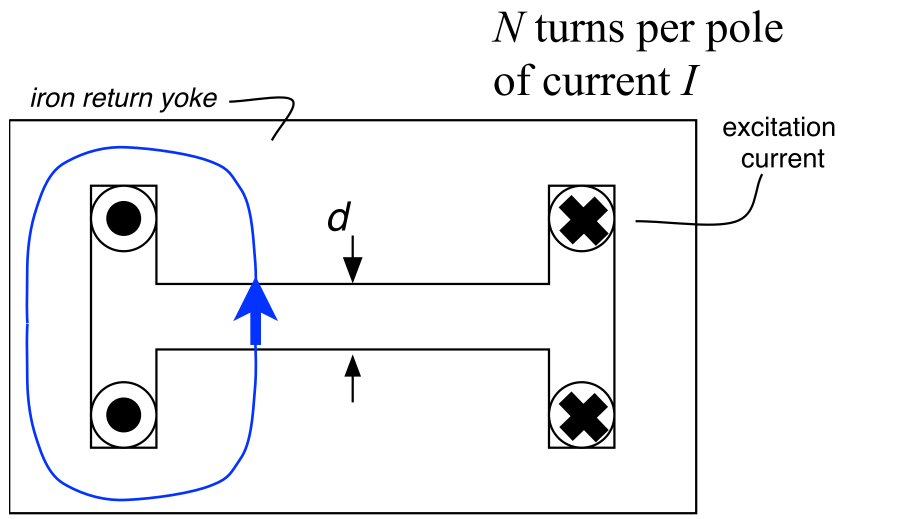
otherwise: $$\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - \left(\frac{1}{1 + W/mc^2}\right)^2}$$

- ions with kinetic energy 12 keV/u: $\beta \approx \sqrt{\frac{2 \cdot 12 \times 10^3}{931 \times 10^6}} = 0.005$
  - (the rest energy of a nucleon is 931 MeV/u)
- A proton with kinetic energy 100 MeV: $\beta \approx 0.428$
- An electron with kinetic energy 1 MeV: $\beta \approx 0.941$

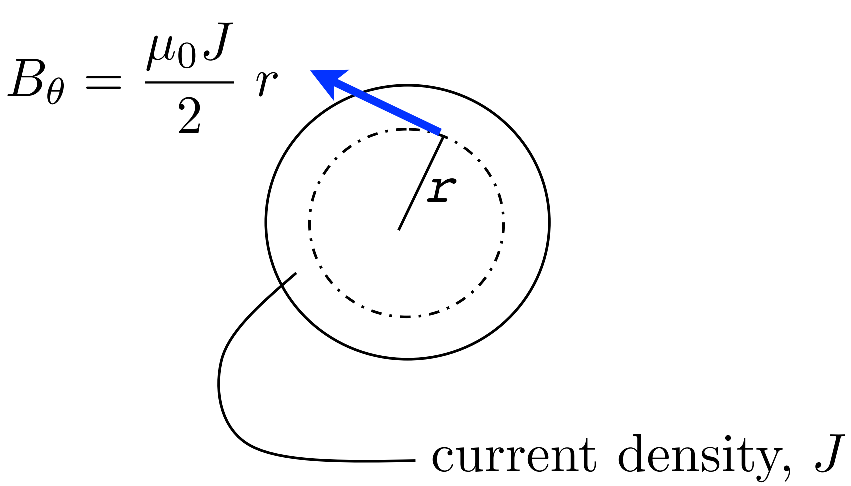
Magnets

- Iron-dominated magnetic fields
  - iron will “saturate” at about 2 Tesla

$$B = \frac{2\mu_0 N \cdot I}{d}$$

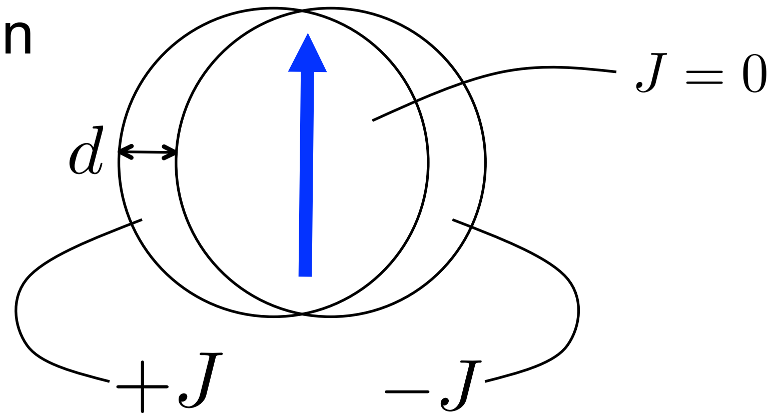


- High-field superconducting magnets
  - field determined by distribution of currents



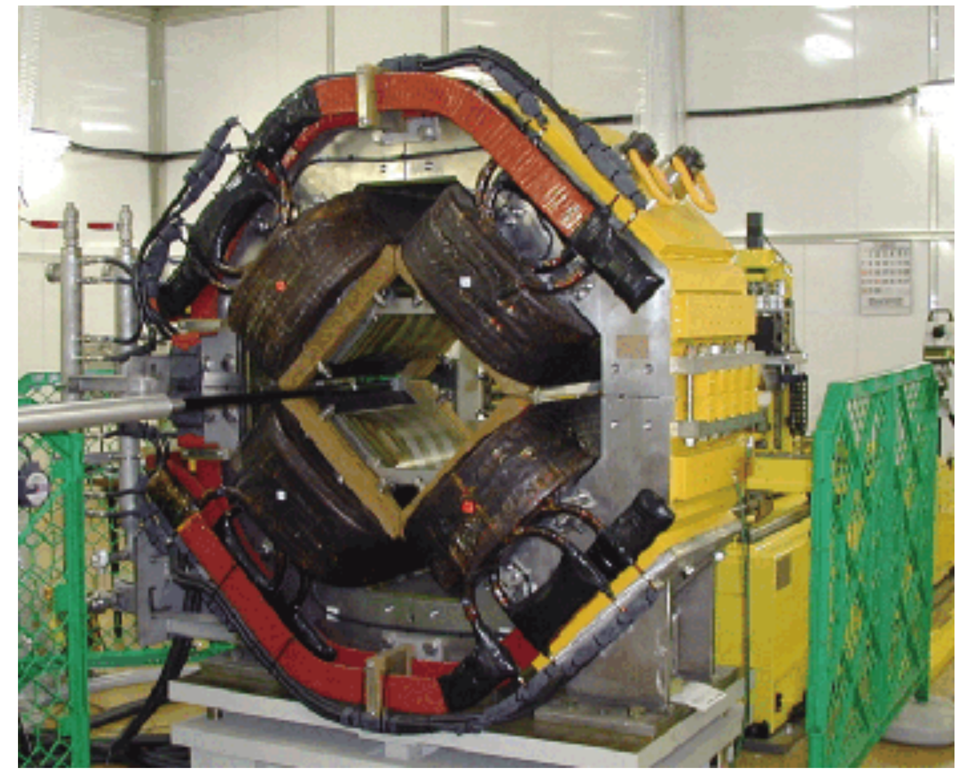
“Cosine-theta” distribution

$$\boxed{B_x = 0, \quad B_y = \frac{\mu_0 J}{2}\, d}$$

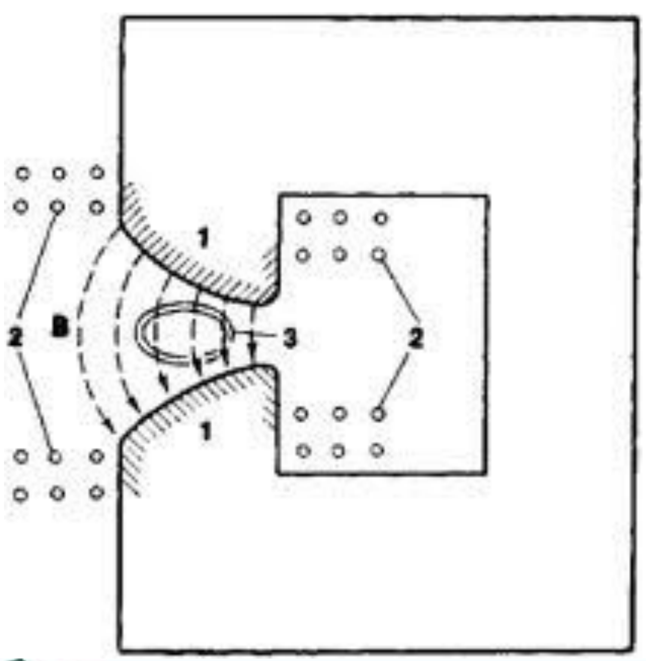


Iron-dominated Magnets

Dipole Magnet, Fermilab

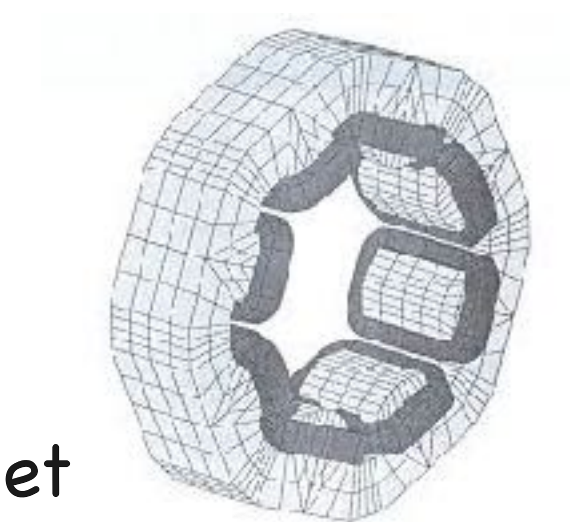


Quadrupole Magnet, J-PARC

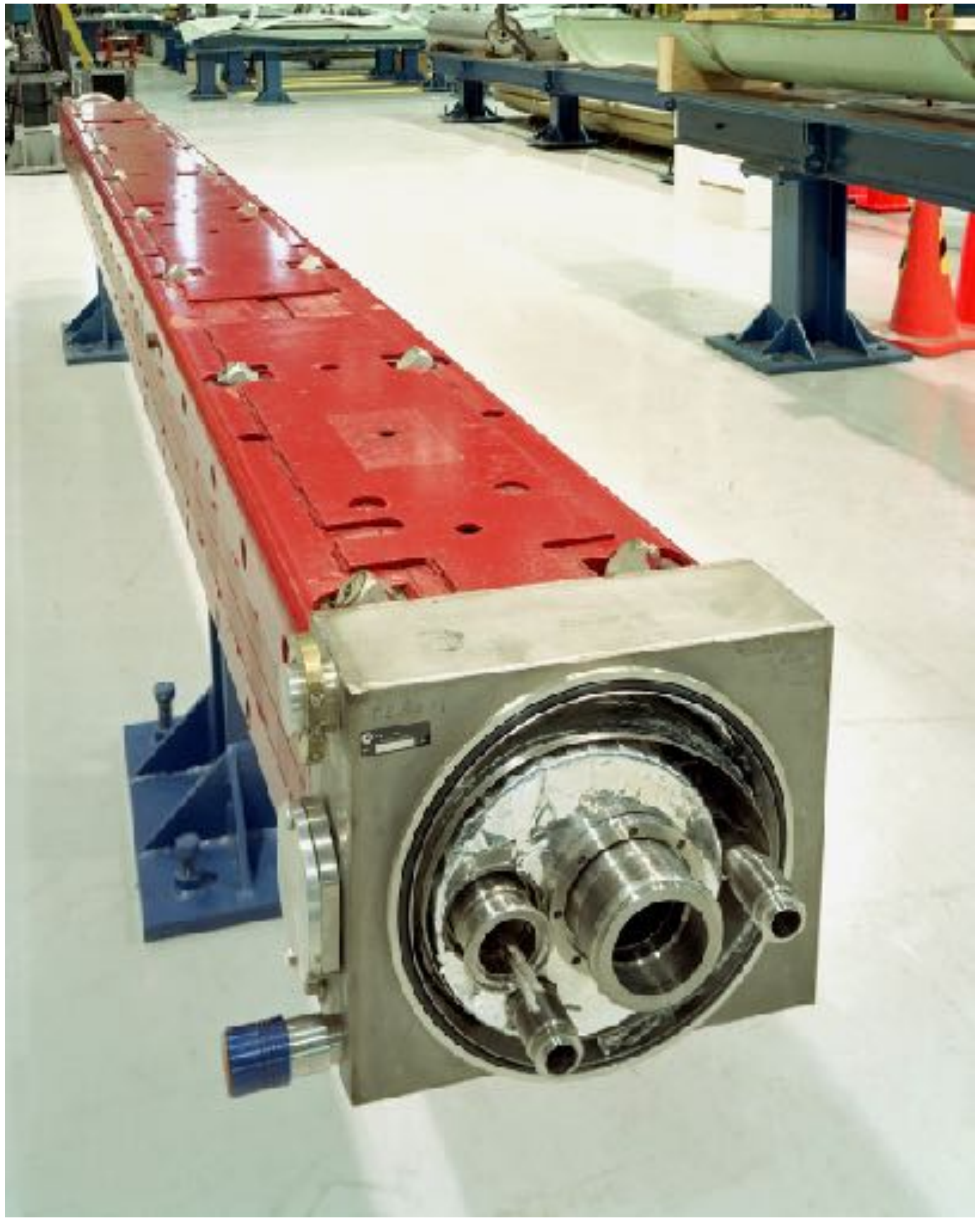


Combined Function
(Weak-focusing)

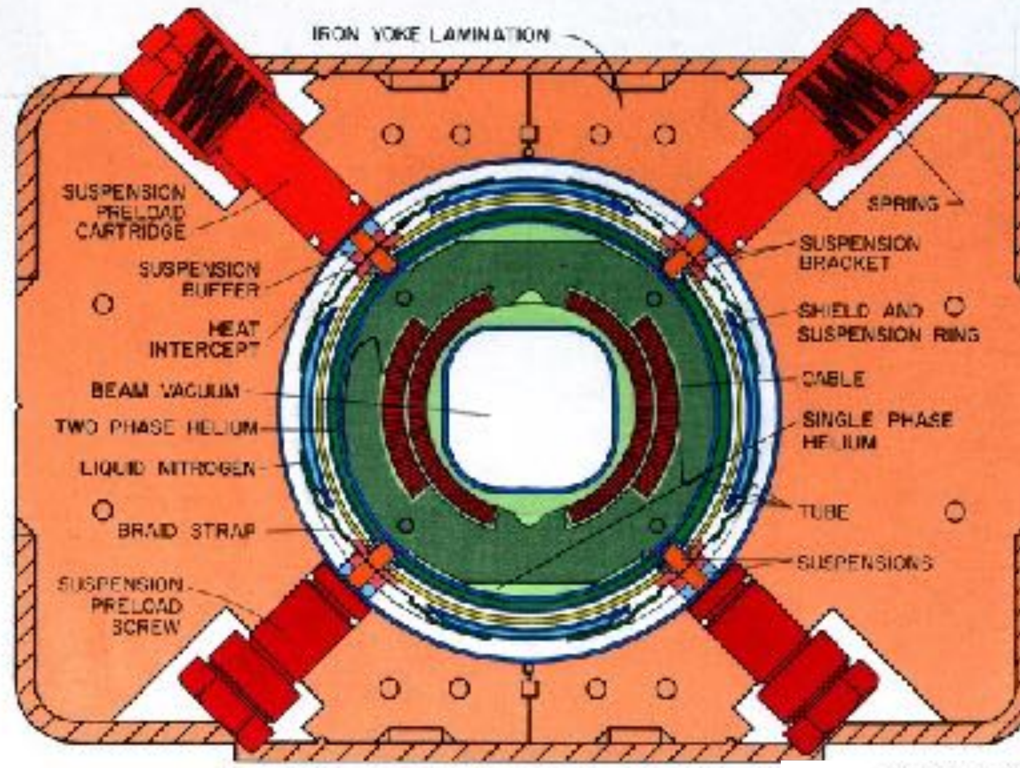
Octupole Magnet

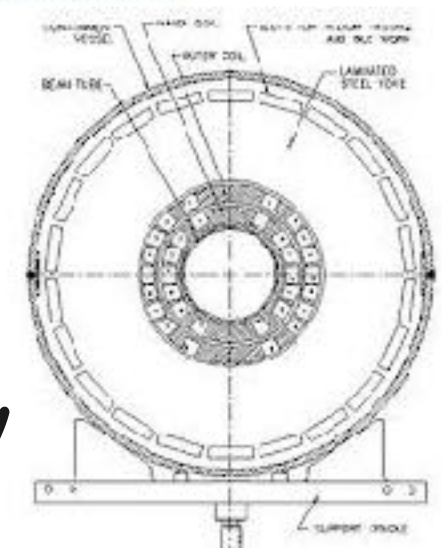


Superconducting Magnet



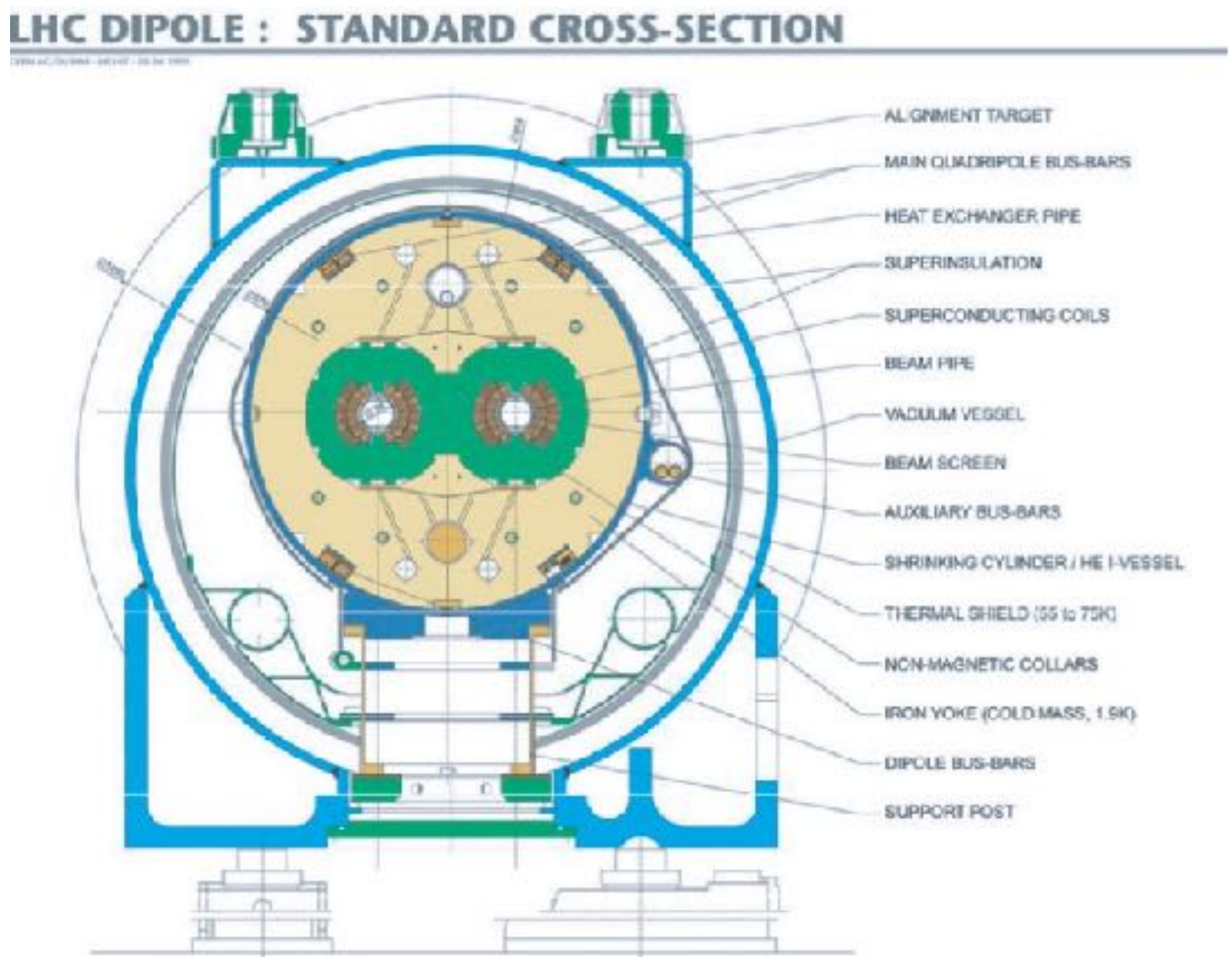
- Tevatron Dipole Magnet

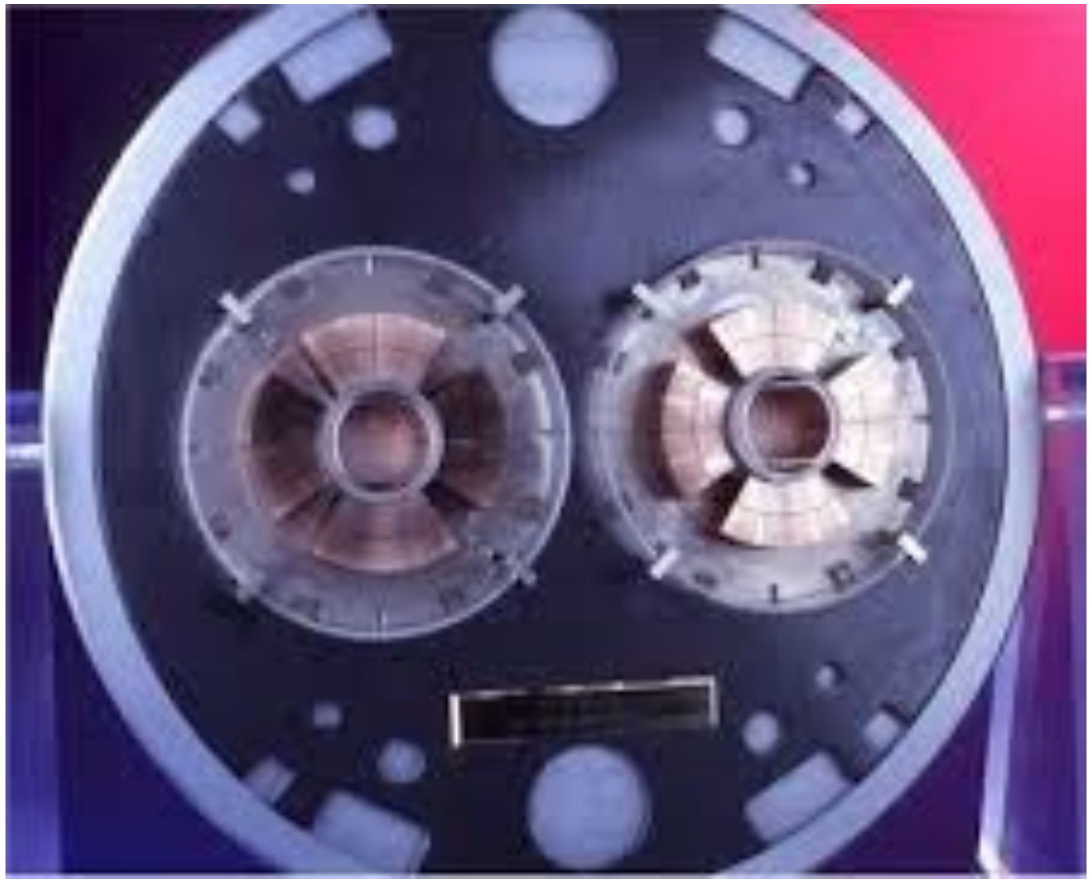




Coil placement is key

Two-in-One

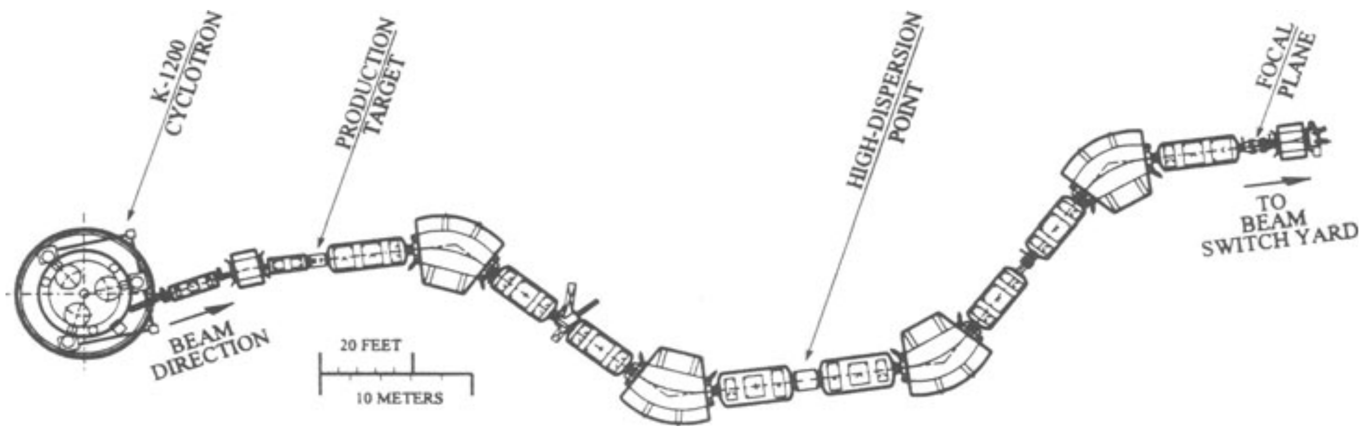


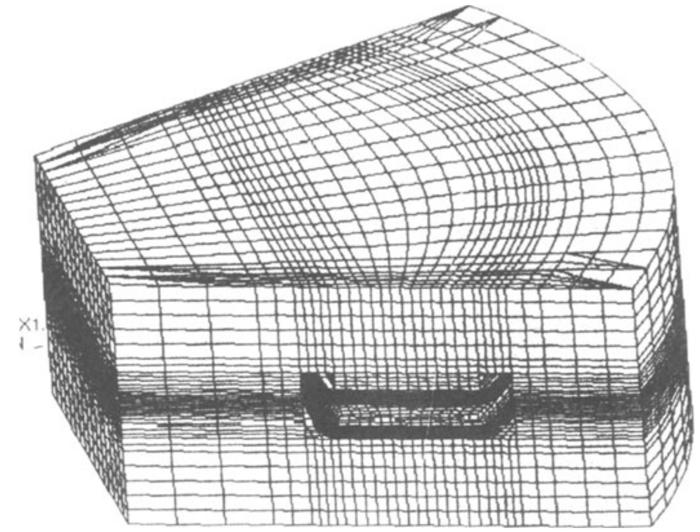


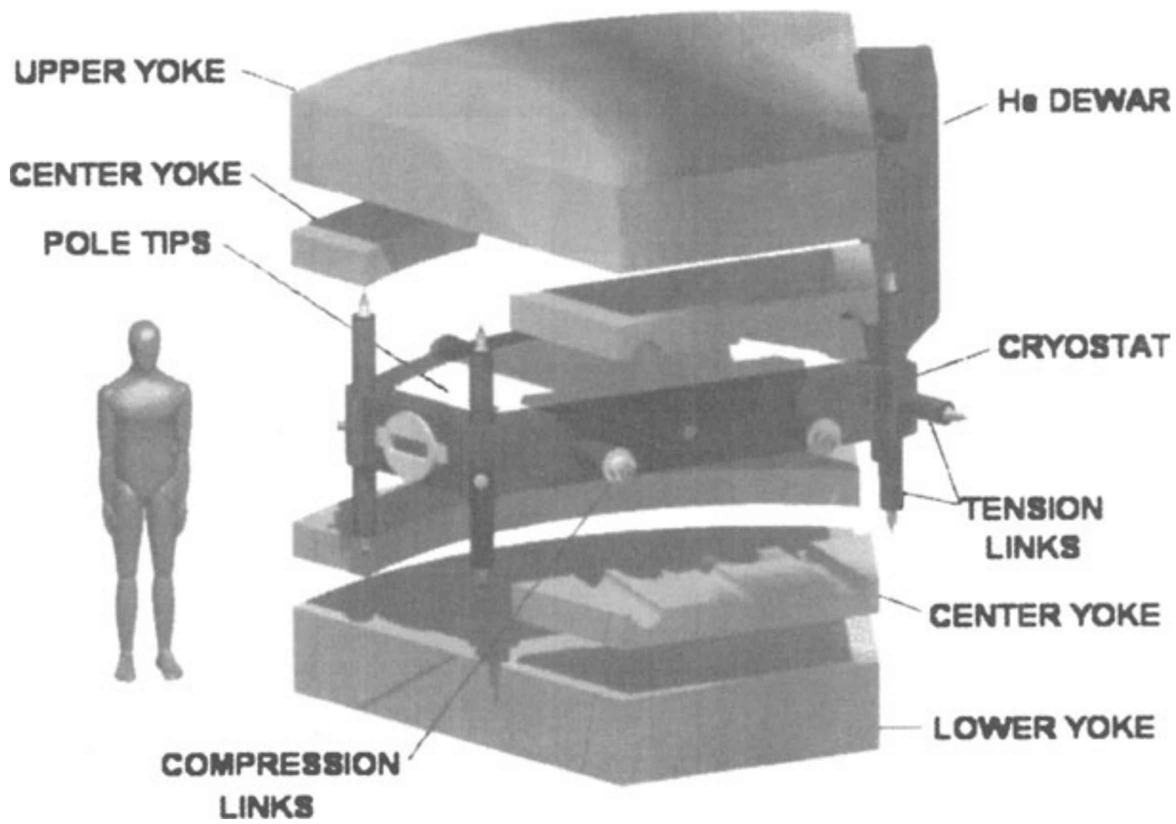
LHC Quadrupole

Super-ferric Magnets

- Ex: The A1900 Dipole at NSCL

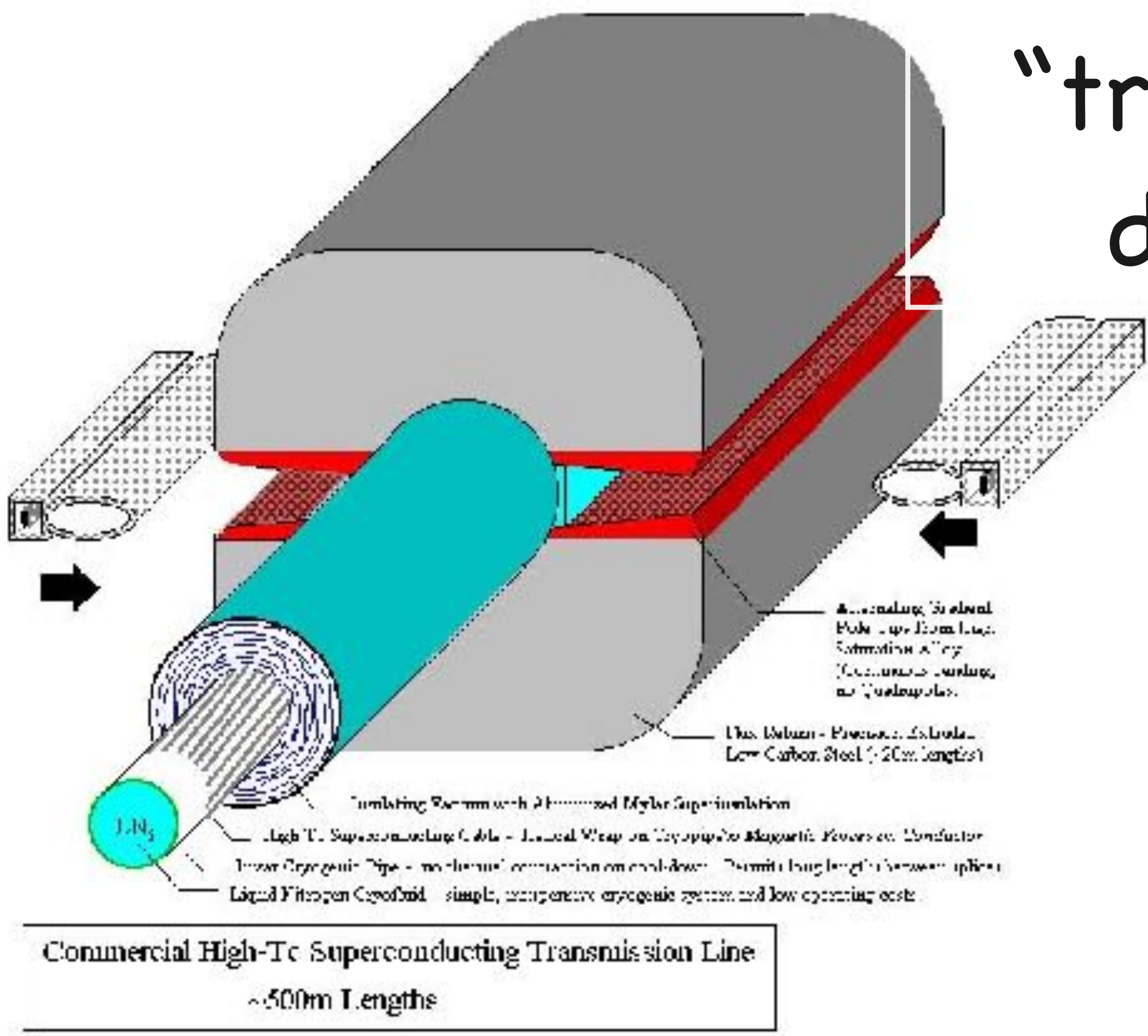




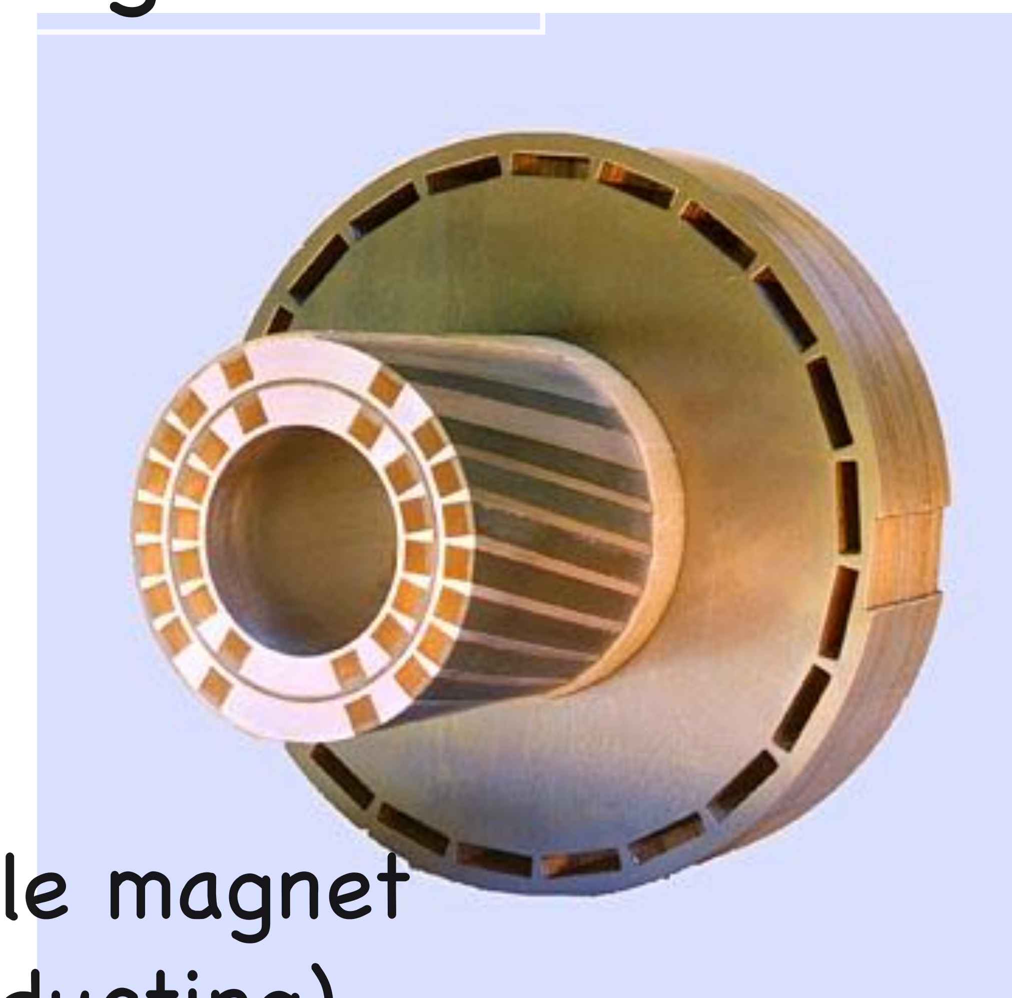


Also…

"Double-C" Twin Bore Transmission Line Magnet

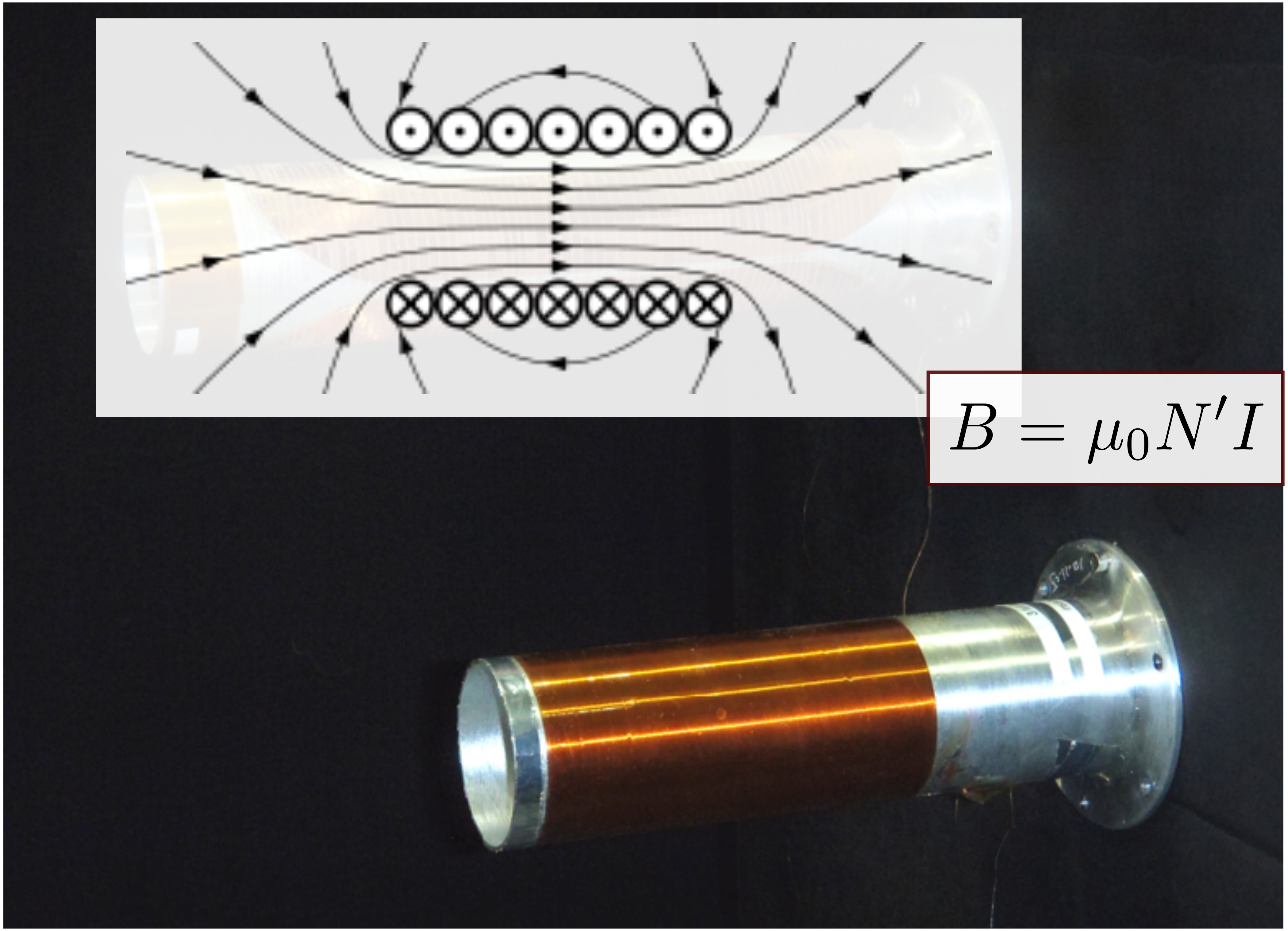


Weak focusing "transmission line" dipole magnet



Helical dipole magnet (superconducting)

Solenoids

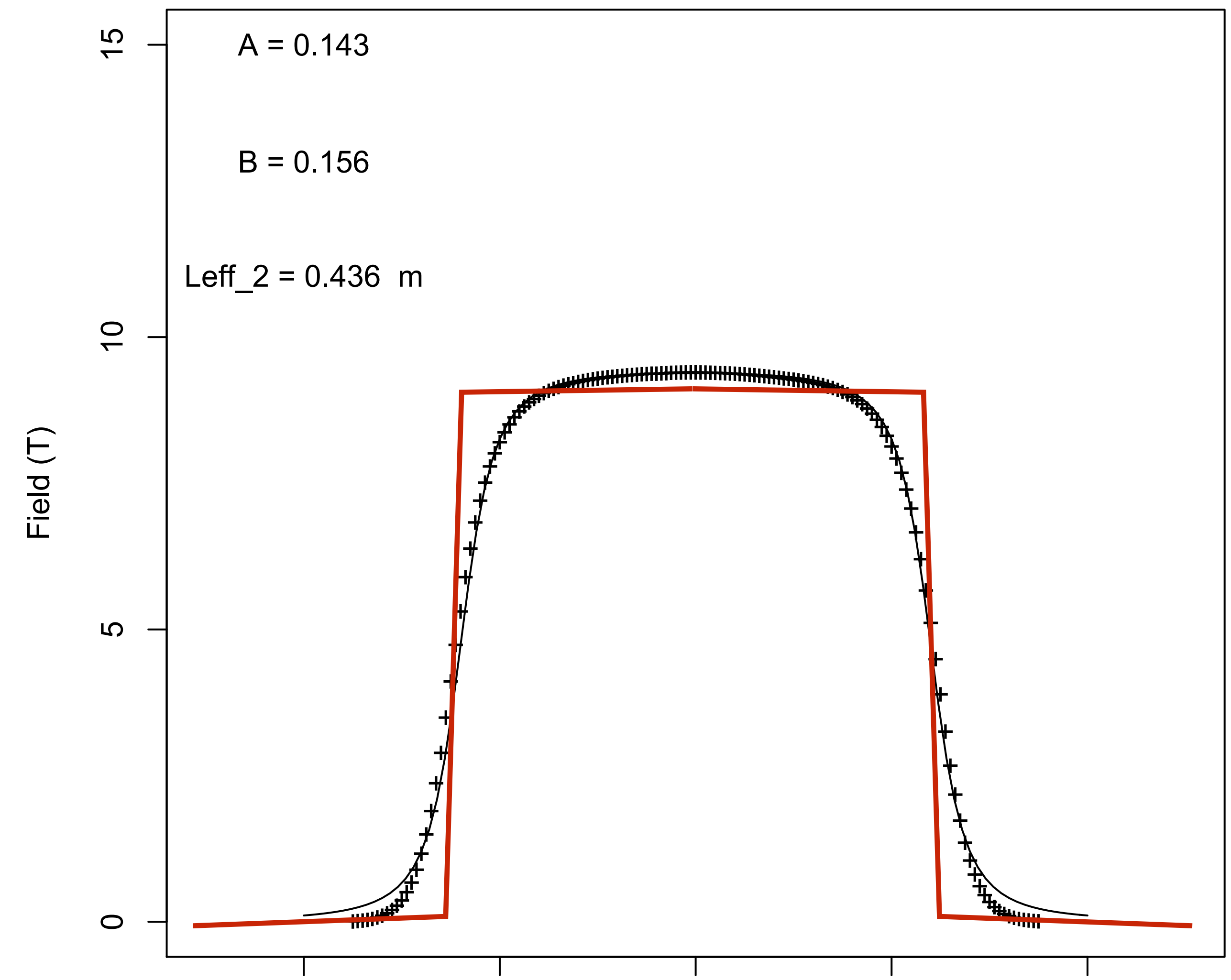


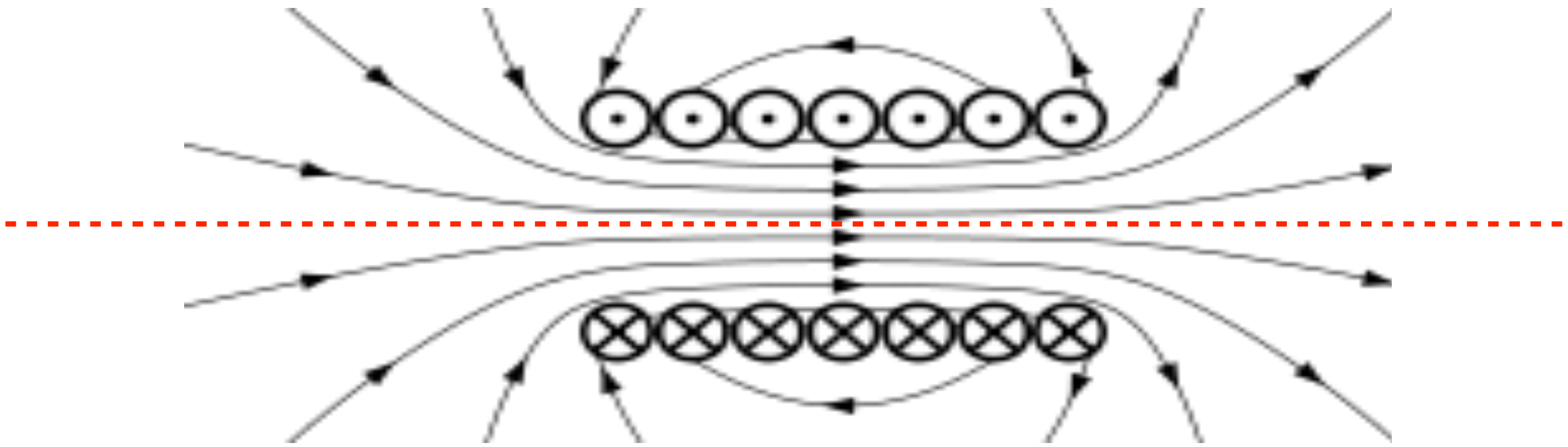
$$B = \mu_0 N' I$$

Effective Length

- Example shown here: Solenoid magnet

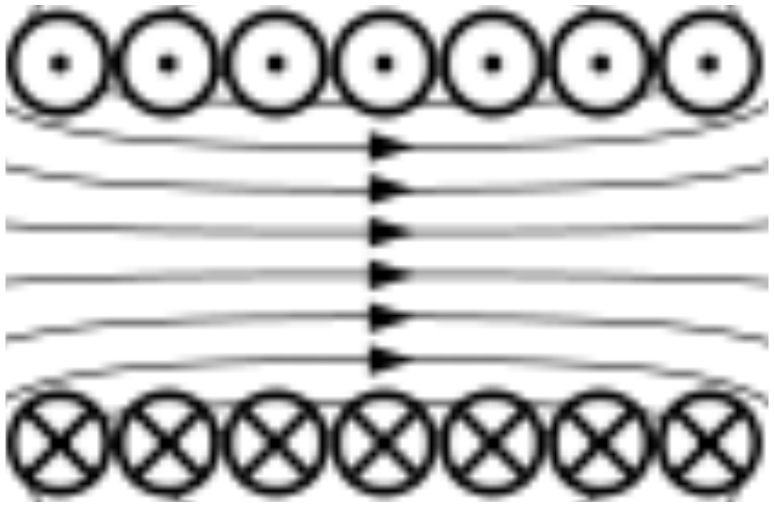
- often think of ends of magnets as “hard edges” and use an “effective length”





Solenoid Focusing

- Solenoid Field



$$B = \mu_0 N' I$$

FRIB linac solenoids ~ 8-9 T

- Particle trajectory in a uniform field

Helical, with radius $a = mv_\perp/(qB_0)$

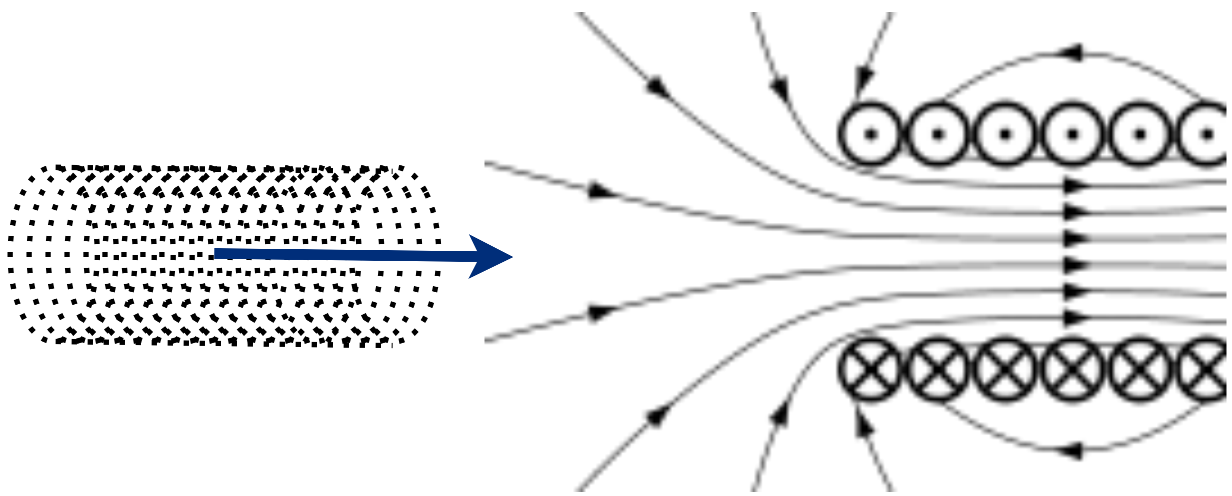
where $v_\perp$ is the velocity perpendicular to $\vec{B}$

$$\omega = \frac{v_\perp}{a} = qB_0/m$$

- So, how does a solenoid "focus"?

Acquisition of Angular Momentum

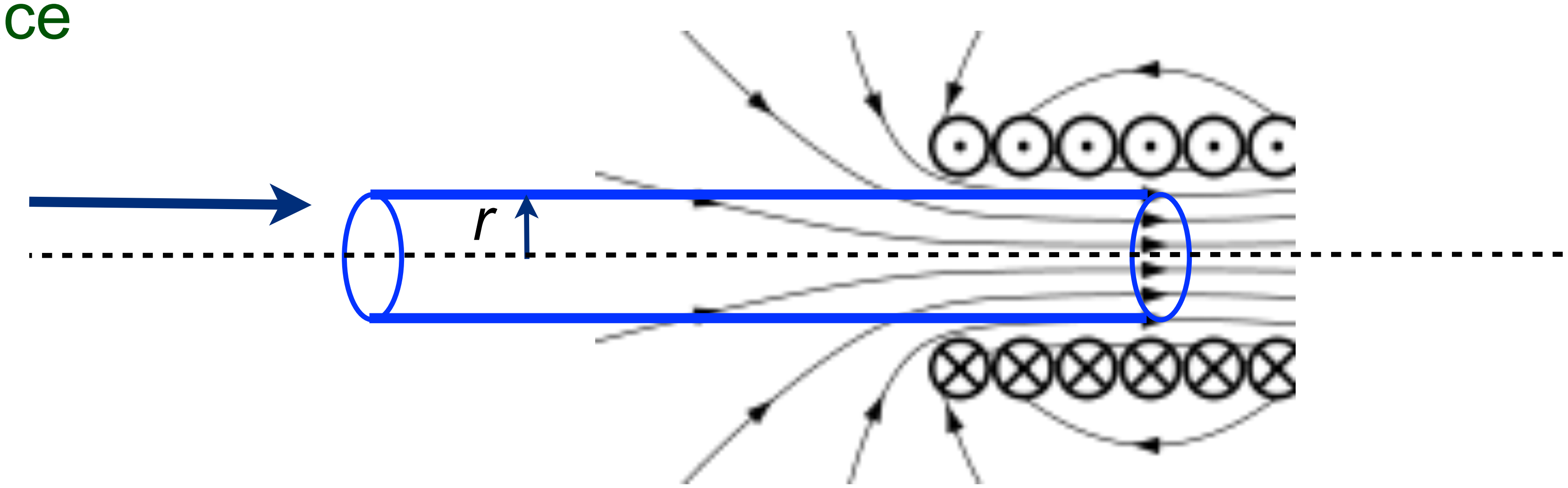
- Imagine particle distribution entering a solenoid magnet, centered on the axis of the magnet, and assume (for now) all trajectories are parallel...



- Treat the "edge" (entrance) of the solenoid as an impulse, but estimate its effect by integrating through the interface

$$\Delta p_\theta \approx q \int_{-\infty}^{0} (\vec{v} \times \vec{B})_\theta dt$$

$$= -\frac{qB_0}{2}\, r$$



Solenoid Focusing

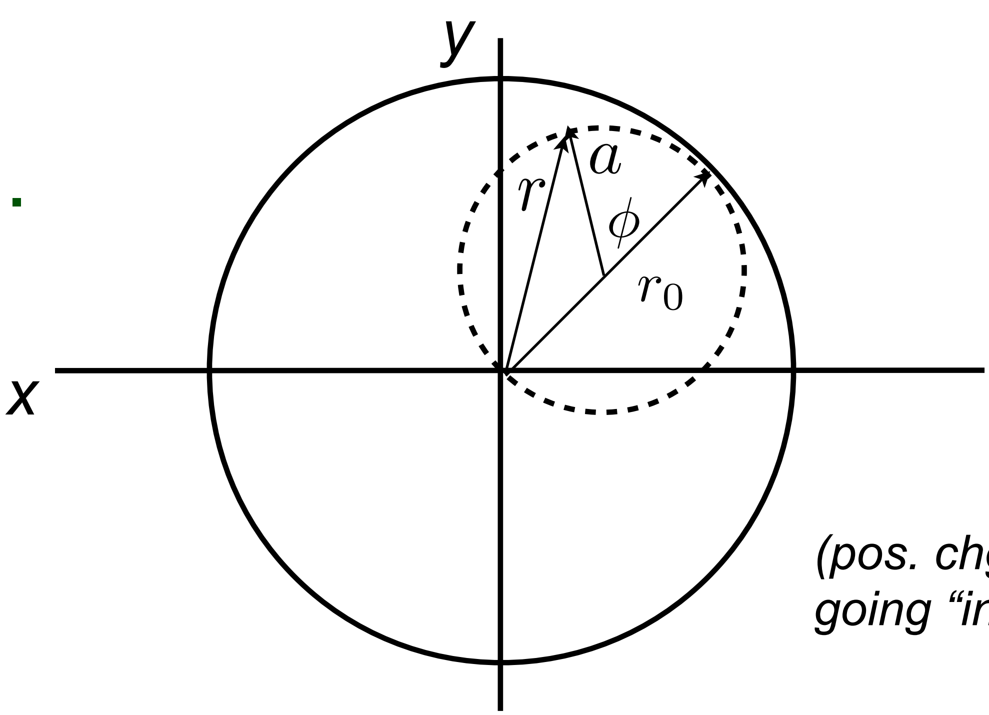
- So, the momentum gained in the theta direction will depend upon its distance from the solenoid axis, and thus its radius of gyration will be given through

$$\omega = qB_0/m = \frac{v_\perp(r)}{(r/2)} \qquad \rightarrow \quad a = r/2$$

- The resulting trajectory will be helical, with radius $a = r_0/2$, and the rotation will advance by an amount

$$\phi(s) = \frac{qB_0}{m}\frac{s}{v}$$

- Thus, ...



(pos. chg. particles going "into" the page)

$$r = r_0 \cos[\phi(s)/2]$$

- Upon exit, the angular momentum of the beam will be removed.

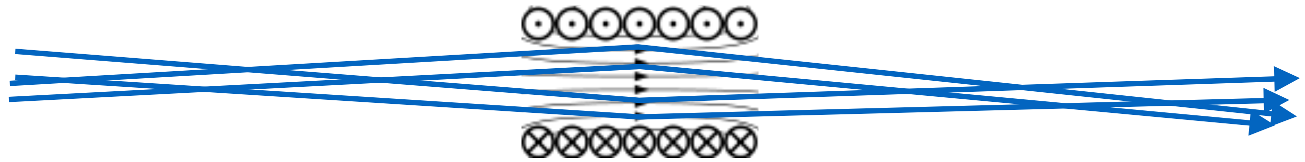
Solenoid Focusing

- We see that the equation of motion of the particle radius is

$$r'' = d^2r/ds^2 = d(r')/ds = -\left(\frac{qB_0}{2mv}\right)^2 r$$

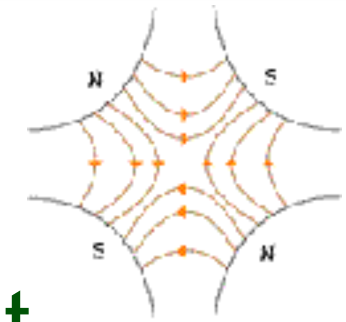
- Thus, a "short" solenoid can be interpreted as a "thin lens" of focal length *f* given by…

$$\frac{1}{f} = \left(\frac{qB_0}{2mv}\right)^2 \ell = \left(\frac{B_0}{2(B\rho)}\right)^2 \ell = \frac{\theta_0^2}{\ell} \qquad \text{where } \theta_0 = B_0\ell/2(B\rho)$$



- Note: "thin" lens $\rightarrow \quad \ell/f = \theta_0^2 \ll 1$
- Use solenoids when *Q* is high and momentum is low.

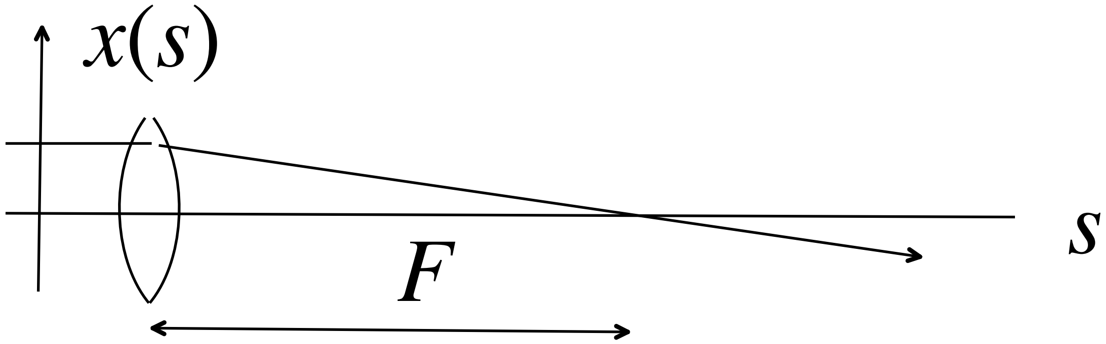
"Thin Lens" Quadrupole



- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics
- Take limit as length *L* --> 0, while *KL* remains finite

$$B' = \frac{\partial B_y}{\partial x}\Big|_{x=0} \qquad KL = \frac{B'L}{B\rho} = \frac{1}{F} \qquad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

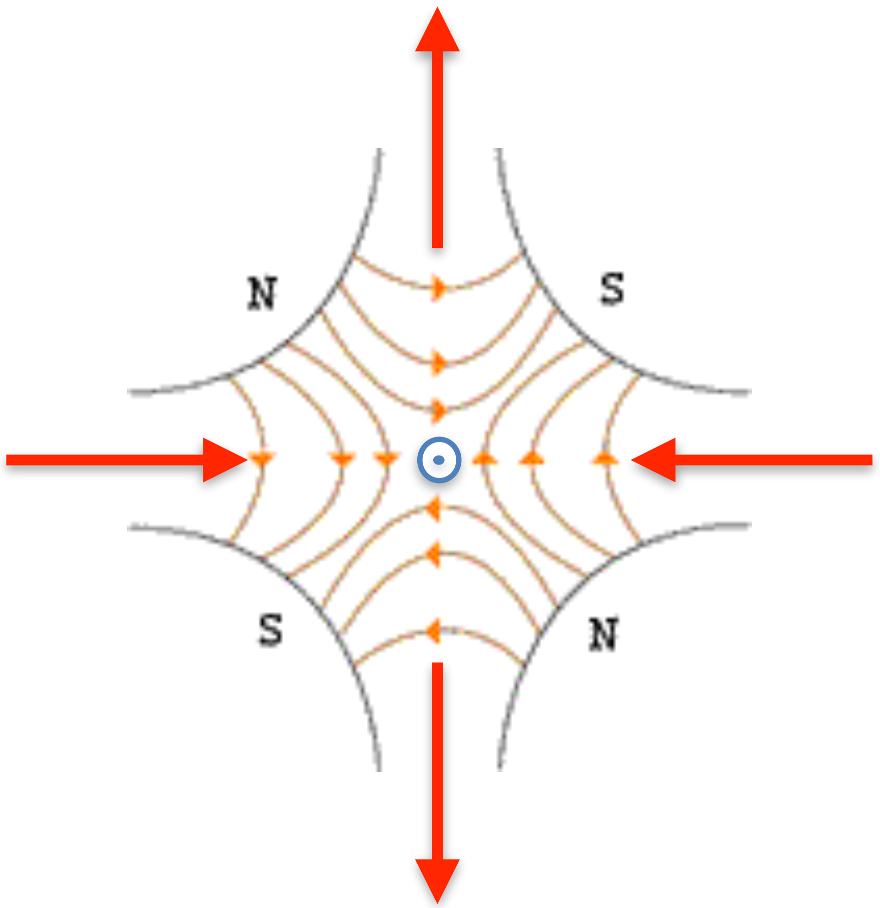
  - (similarly, for defocusing quadrupole)



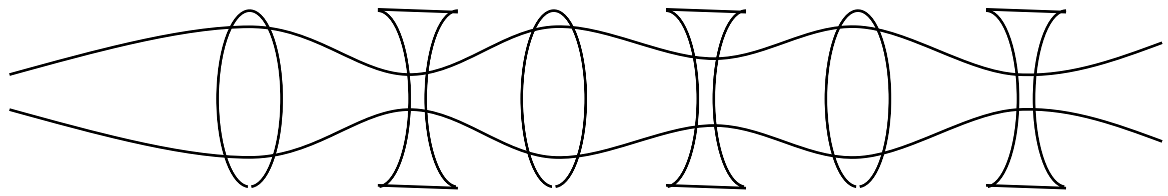
- Valid approx., if *F* >> *L*

Strong (Alternating Gradient) Focusing

- Think of standard focusing scheme as alternating system of focusing and defocusing lenses (today, use quadrupole magnets)

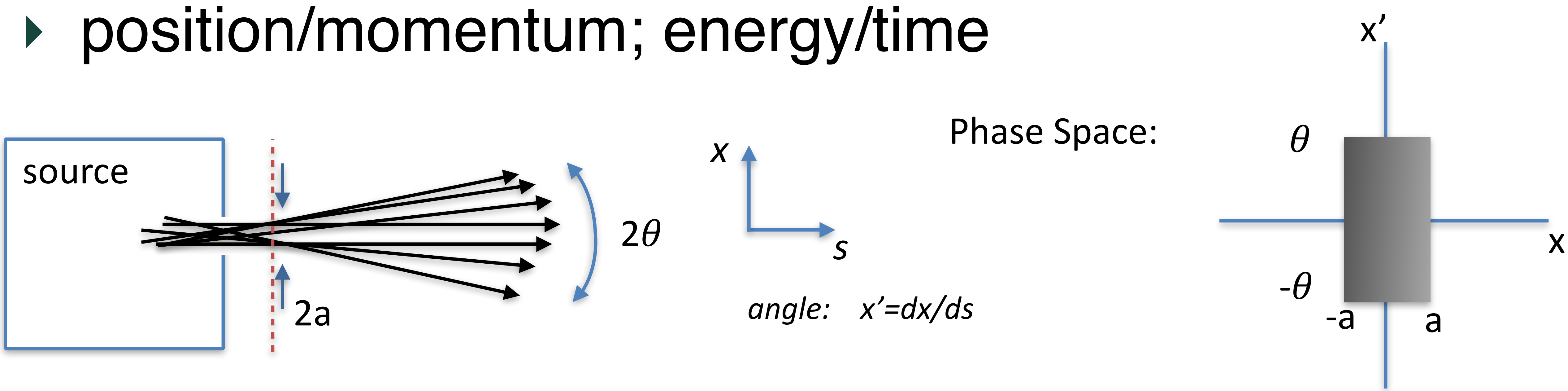


- Quadrupole will focus in one transverse plane, but defocus in other; if alternate, can have net focusing in both
  - alternating gradients:



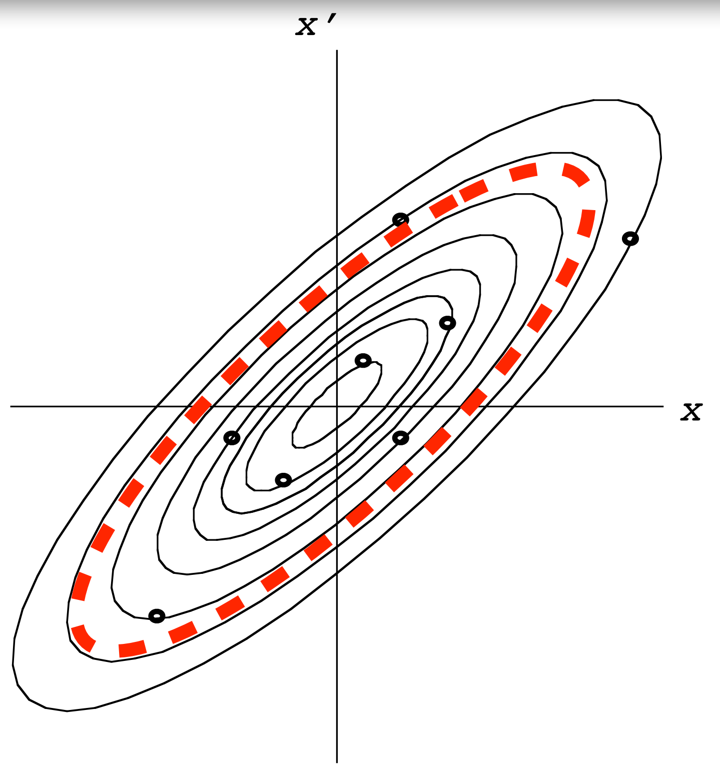
Particle Beams

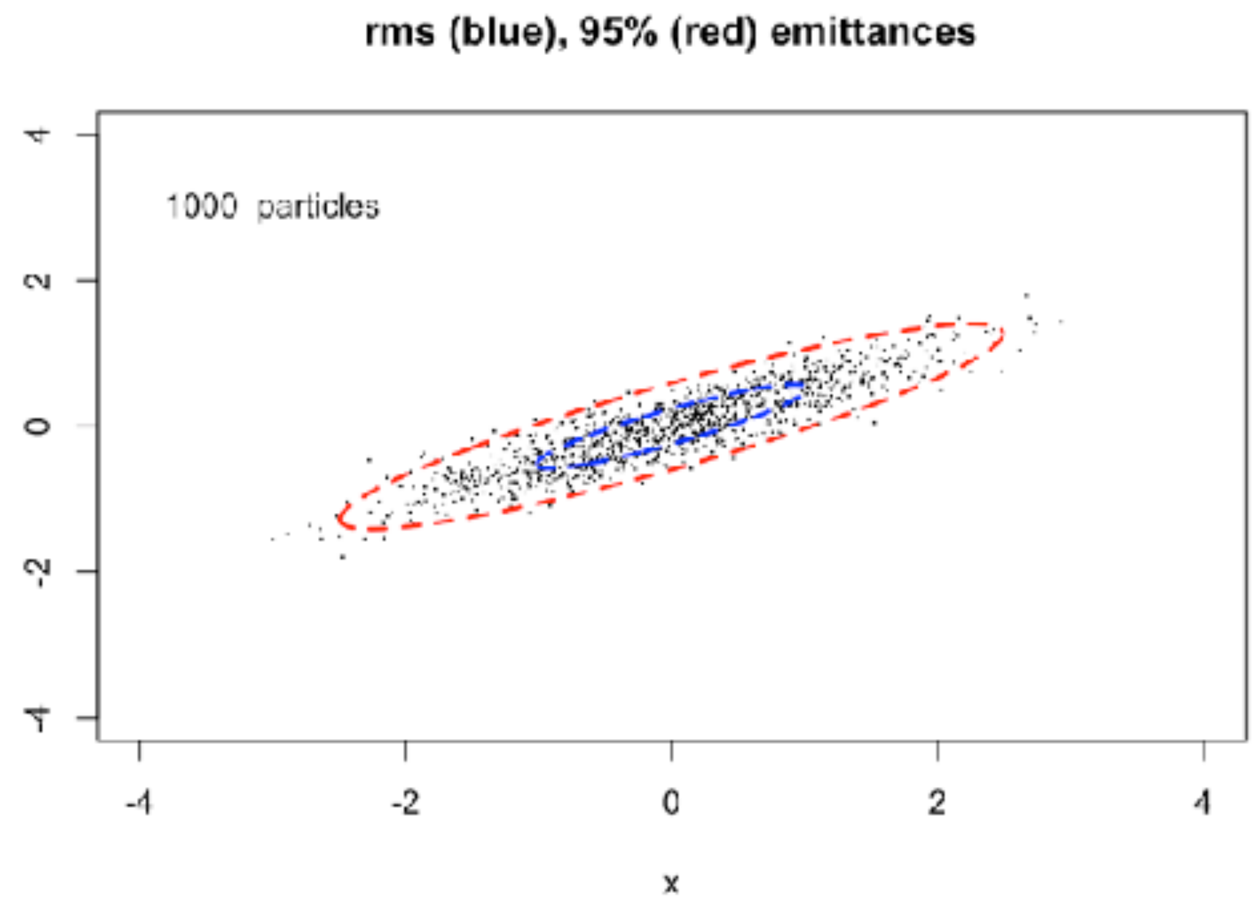
- Not just one particle, but a "bunch" of particles
- Finite spread in particle properties
  - energy / momentum spread
  - position / direction spread
- Characterization in terms of "phase space"
- Adiabatic invariance of phase space variables
  - position/momentum; energy/time



Beam Emittance

- Phase space area which contains a certain fraction of the beam particles
- Popular Choices:
  - 95%
  - 39%
  - 15%

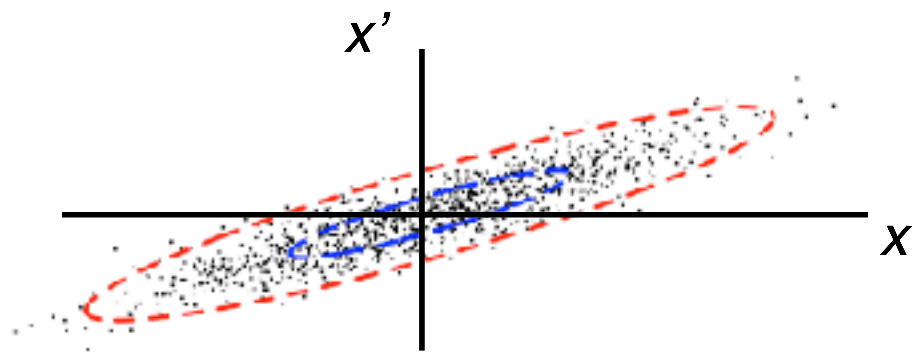




- For each particle, the "size" of its ellipse is determined by the initial conditions (x_0, x'_0) for that particle; the "shape" of the ellipse is determined by the arrangement of focusing elements.

Emittance in Terms of Moments

- Considering the general equation of an ellipse, the area enclosed by the ellipse — the emittance — is related to the coefficients by:



$$ax^2 + bxx' + cx'^2 = 1$$

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define scaled quantities from our distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\boxed{\epsilon = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}}$$

α, β, γ collectively are called the *Courant-Snyder* parameters, or *Twiss* parameters

the “rms emittance”

So, equation of the **blue** curve above:

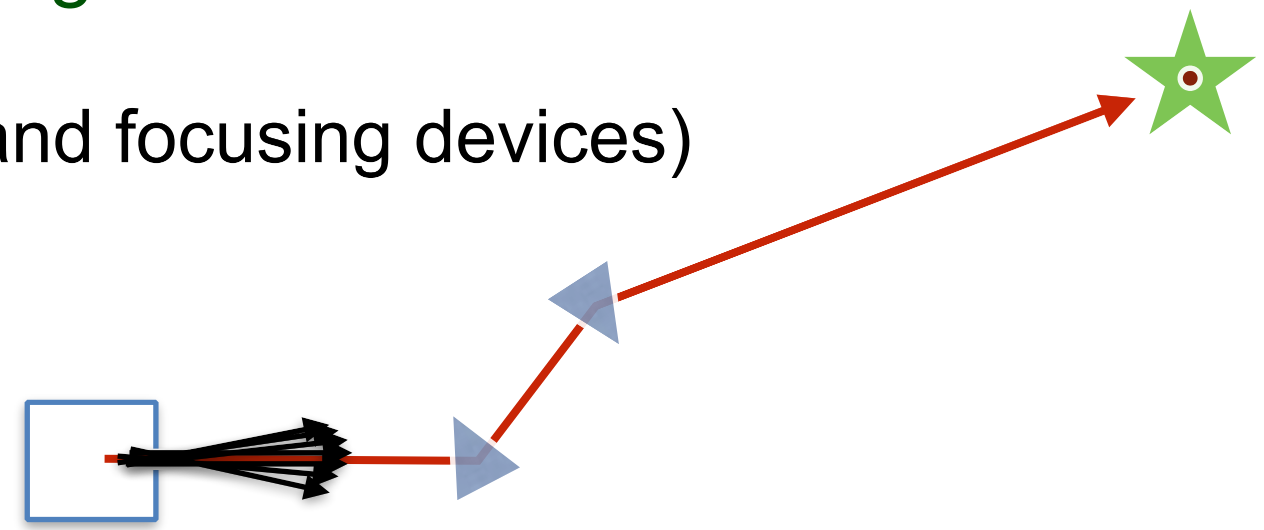
$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon/\pi$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

Essential Beam Transport and Focusing

- Can imagine using a section of finite length containing pure uniform magnetic field to bend a charged particle's trajectory through a portion of a circular arc, thus steering it in a new direction. An arrangement of such magnets can thus be used to guide an "ideal" particle from one point to another
- However, most (all?) particles are NOT ideal! Hence, as particles drift away from the ideal trajectory, we wish to guide them (using quadrupole magnets or solenoids) back toward the ideal.
- Will use discrete electromagnets of finite length and assume a linear relationship between a particle's exit trajectory to its entrance trajectory, depending upon the strength of the magnetic field

  • (similar rules for electrostatic bending and focusing devices)



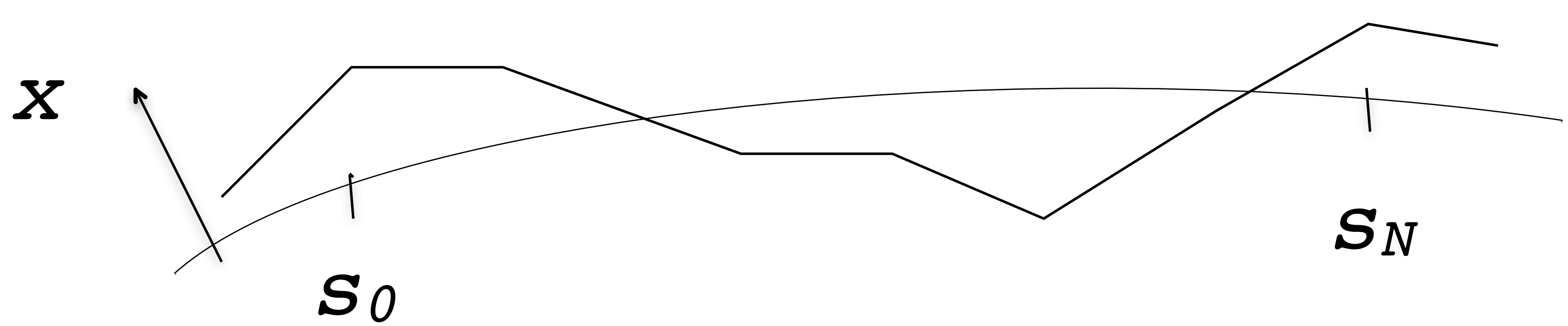
Linear Optics

- Let *x* be the transverse (horizontal, say) displacement of a particle from the ideal beam trajectory. Let the angle it makes to the ideal trajectory be *x'* = *dx*/*ds*, where *s* is the distance along the ideal trajectory. Transport through a magnetic element is then described by a matrix *M*, such that

$$\vec{X} = M\vec{X}_0 \qquad \vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

- An arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication for elements all along the beam line…

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Piecewise Method -- Matrix Formalism

- Write solution to each "piece" of the beam transport system in matrix form
  - for each piece, assume K = const. from s=0 to s=L

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

free space/drift, or bending magnet

focusing (quadrupole) field

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

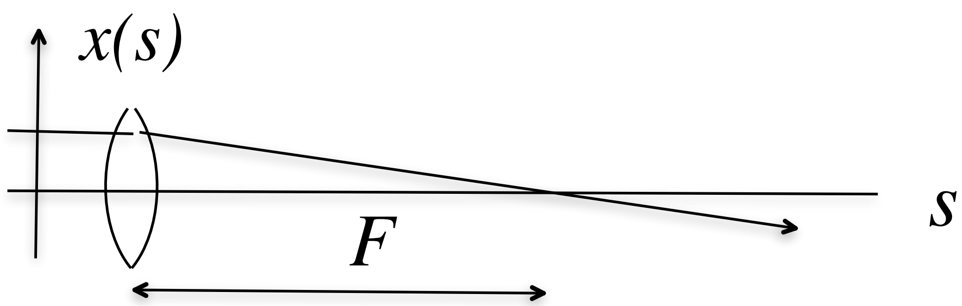
- $K < 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

defocusing (quadrupole) field

“Thin Lens” Quadrupole

- If quadrupole magnet is short enough, particle’s offset through the quad does not change by much, but the slope of the trajectory does -- acts like a “thin lens” in geometrical optics



- Take limit as L --> 0, while KL remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

- (similarly, for defocusing quadrupole)
- valid approximation, if F >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

TRANSPORT of Beam Moments

- Transport of particle state vector downstream from position 0

$$\vec{X} = \left(\begin{array}{c} x \\ x' \end{array} \right) \qquad \vec{X} = M\vec{X}_0$$

- Create a “covariance matrix” of the resulting vector…

$$\vec{X}\vec{X}^T = \left(\begin{array}{cc} x^2 & xx' \\ x'x & x'^2 \end{array} \right) = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \left(\begin{array}{cc} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{array} \right)$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

- So, since

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$

- where

$$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

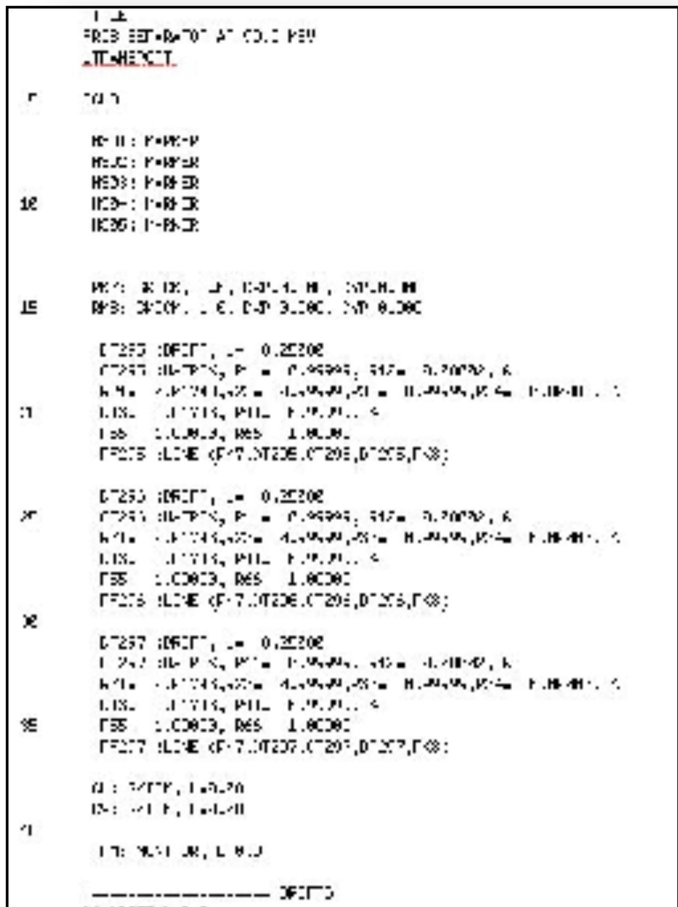
$$\boxed{K = M\ K_0\ M^T}$$

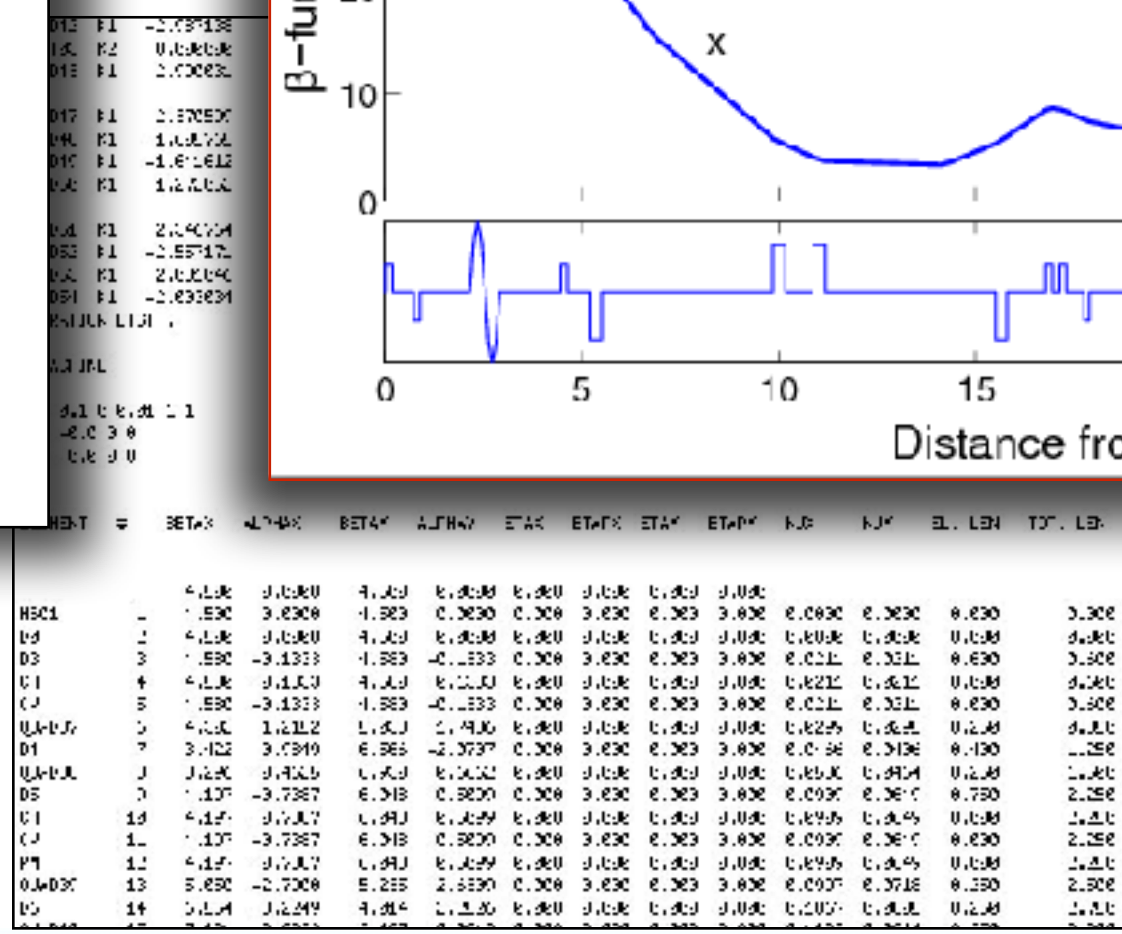
- If know matrices *M*, then can "transport" beam parameters from one point to any point downstream, which determines beam distribution along the way.

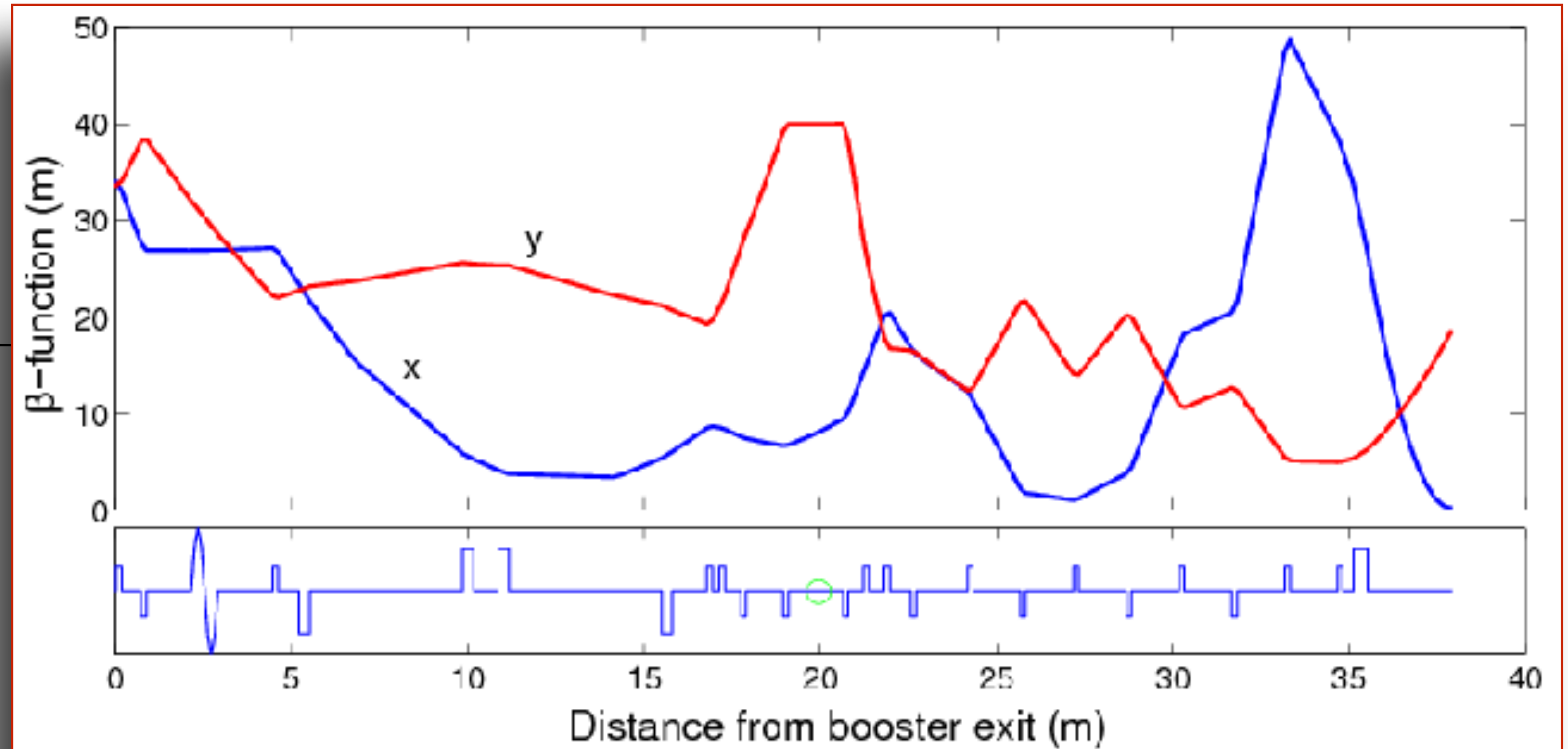
$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \rightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD, TRACE, TRACE3D, COSY, SYNCH, CHEF, many more ...







Hill's Equation — Analytical Solution

- For an individual particle in our distribution, the equation of motion is of the form (Hill's Equation):

$$x'' + K(s)x = 0 \qquad\qquad K = \frac{\partial B_y/\partial x}{B\rho} = \frac{B'}{B\rho}$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, *s*.

- So, assume solution is sinusoidal, with a phase which advances as a function of location *s*; also assume amplitude is modulated by a function which also depends upon *s*:

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

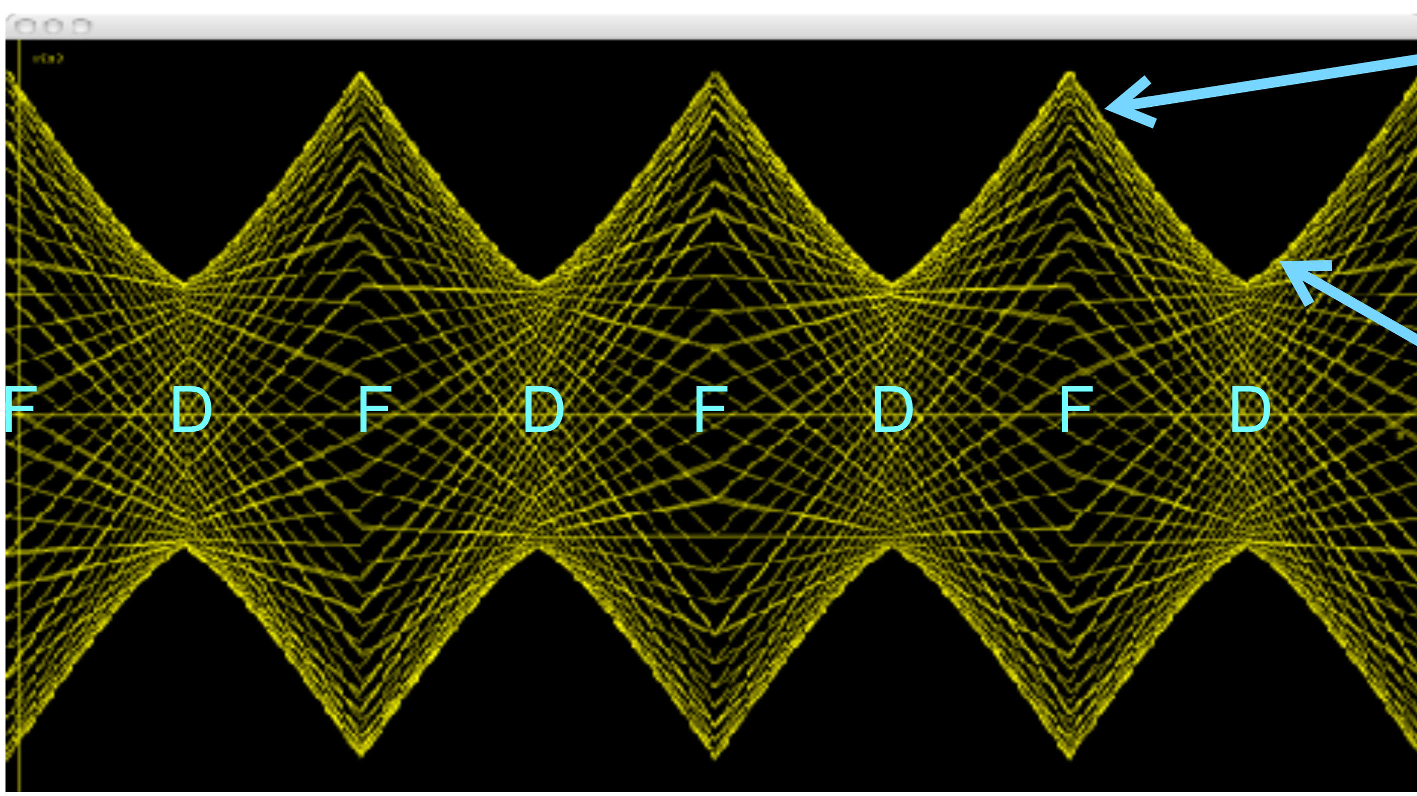
- We find that β is Courant-Snyder parameter found earlier

and $d\psi/ds = 1/\beta(s)$

The Amplitude Function, β

$$x(s) = A\sqrt{\beta(s)}\,\sin[\psi(s) + \delta]$$

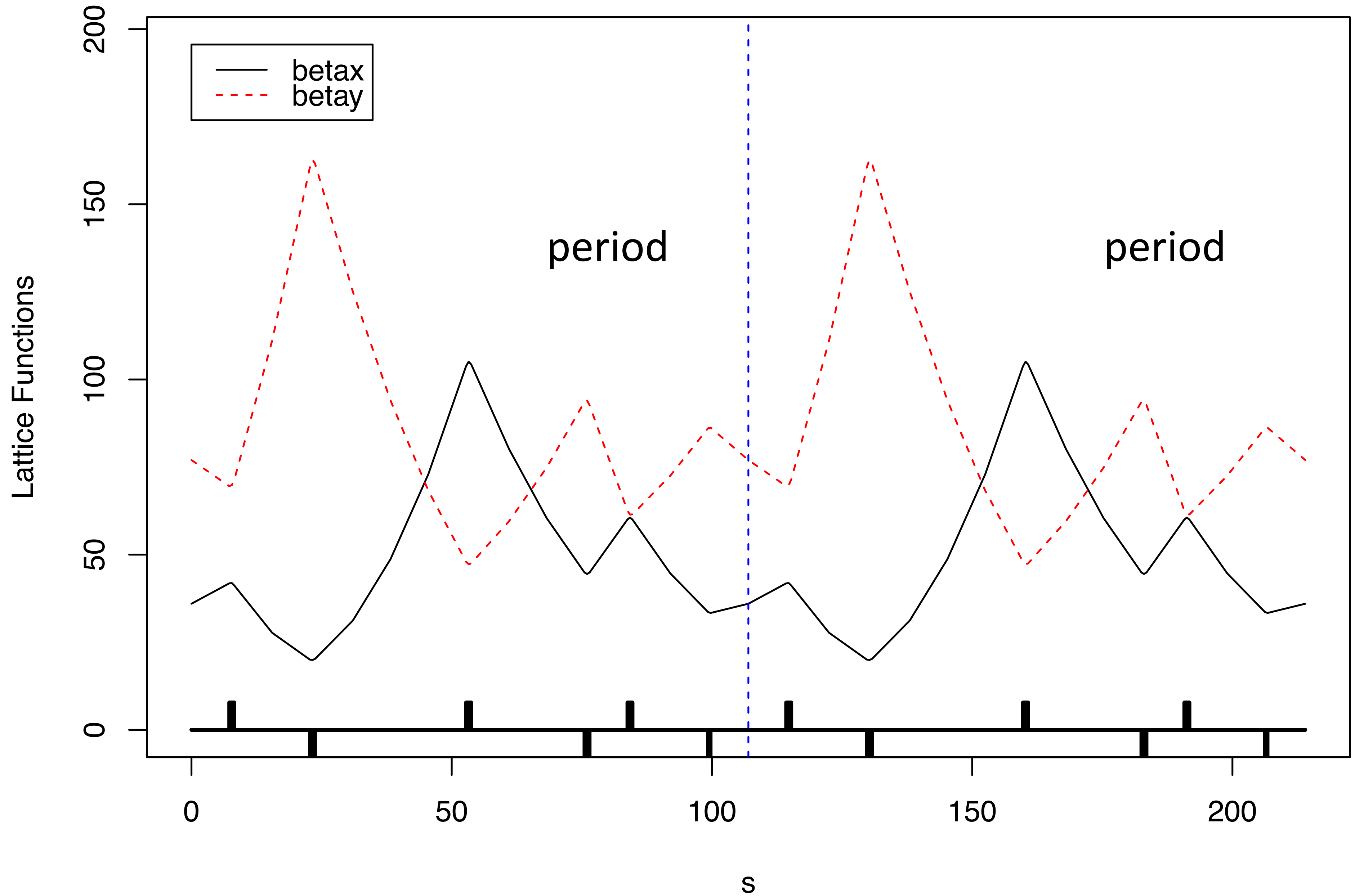
- $d\psi/ds = 1/\beta$, hence β is a local "wavelength". As such, it might have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant A must have units of $m^{1/2}$, and it must be numerically small.
- In conjunction with earlier discussion, πA^2 = single-particle emittance

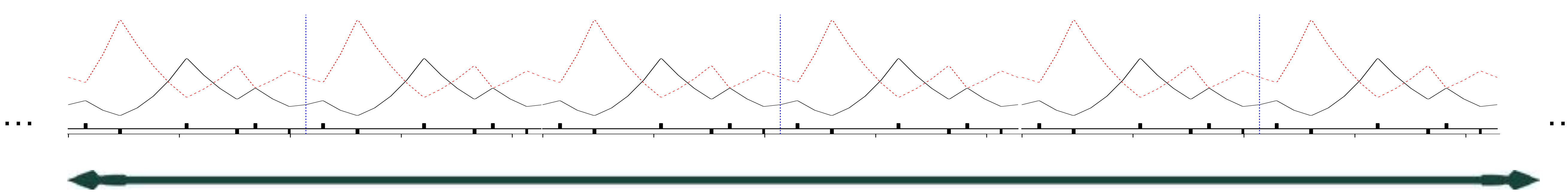


Higher β --
smaller phase advance
larger beam size

Lower β --
greater phase advance
smaller beam size

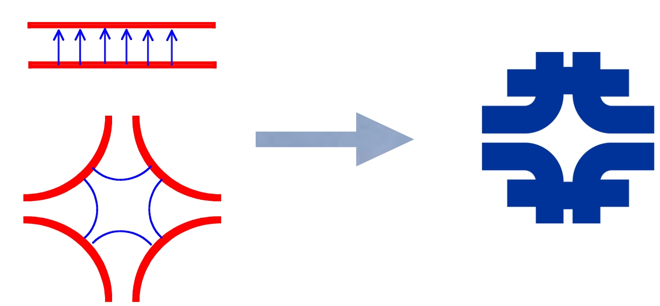
Ex: Periodic Distribution of Quadrupoles





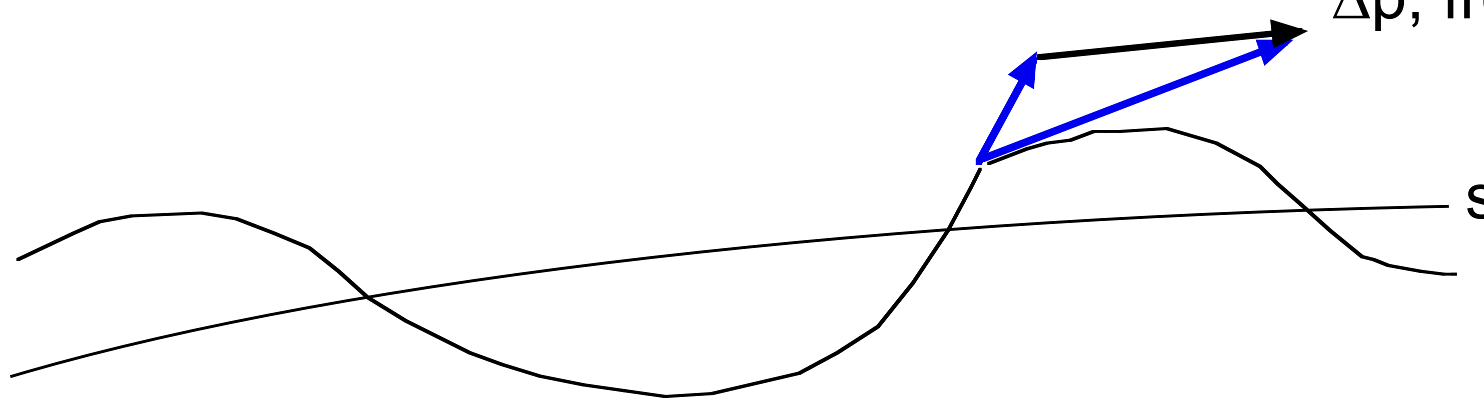
Example: The Fermilab "Main Injector"

"separated function"
first used at Fermilab



Adiabatic Damping from Acceleration

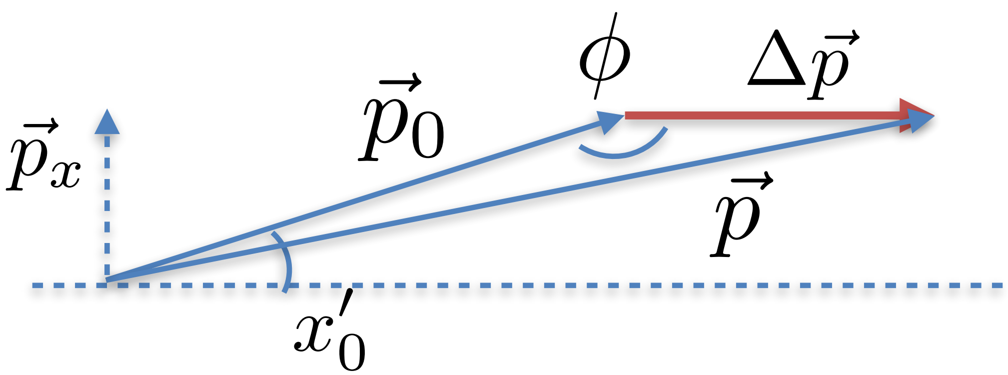
- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates x-x' are not canonical conjugates, but x-p_x are; thus, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration

details…



Note: assuming that ALL particles receive the same *Δp* from the cavity

$$x' = \frac{p_x}{p} = \frac{p_x}{\sqrt{p_0^2 + \Delta p^2 - 2\Delta p\, p_0 \cos\phi}} = \frac{p_x}{p_0}\left(1 - \frac{\Delta p}{p_0} + \dots\right) \approx x_0'\left(1 - \frac{\Delta p}{p_0}\right)$$

Note: particles at peak of their betatron oscillation will have little/no change in *x'*, while particles with large transverse angles will have their *x'* affected most

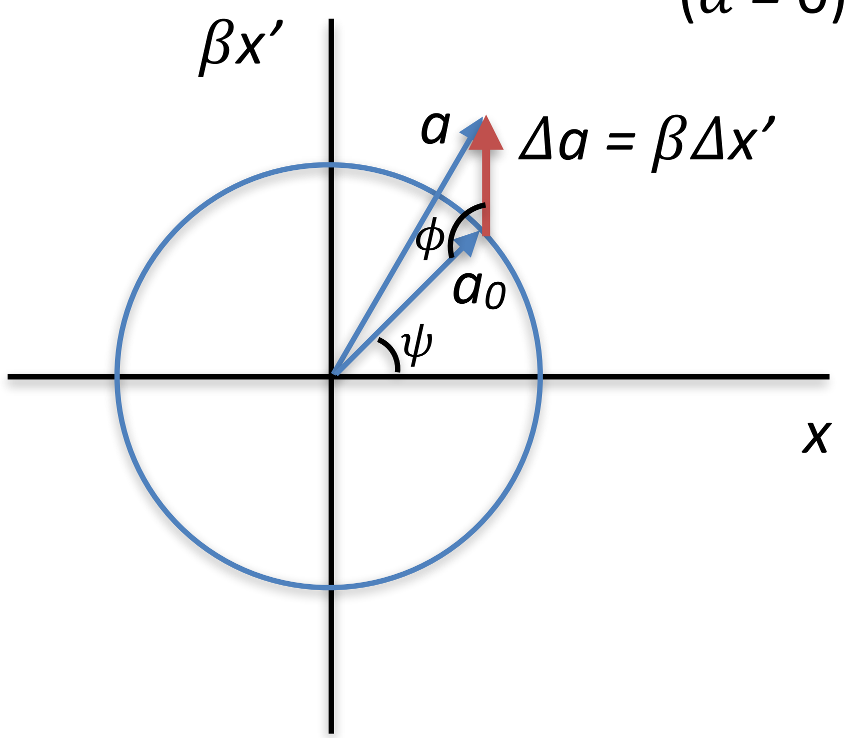
$$\Longrightarrow \quad \Delta x' = -x_0' \frac{\Delta p}{p_0}$$

Adiabatic Damping from Acceleration

details…

Now assuming that the incremental change in momentum is small compared to the overall momentum, and that the changes occur gradually on time scales large compared to those of the betatron motion…

(α = 0)



$$\boxed{\Delta x' = -x_0' \frac{\Delta p}{p_0}}$$

$$\begin{aligned}
a^2 &= a_0^2 + \Delta a^2 - 2\Delta a\, a_0 \cos\phi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\, a_0 \sin\psi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\, \beta\, x_0' \\
&= a_0^2 + (-\beta x_0' \frac{\Delta p}{p_0})^2 + 2(-\beta x_0' \frac{\Delta p}{p_0})\beta x_0' \\
&= a_0^2 + (\beta x_0')^2 \cancel{(\frac{\Delta p}{p_0})^2} - 2(\beta x_0')^2 \frac{\Delta p}{p_0}
\end{aligned}$$

$$\Longrightarrow \quad \langle a^2 \rangle = \langle a_0^2 \rangle - 2\langle(\beta x_0')^2\rangle \frac{\Delta p}{p_0}$$

Note: $2\langle(\beta x')^2\rangle = 2\langle x^2 \rangle = \langle a^2 \rangle$

So, $\Delta\langle a^2 \rangle = -\langle a^2 \rangle \frac{\Delta p}{p}$ $\qquad \frac{\Delta\epsilon}{\epsilon} = -\frac{\Delta p}{p}$

$$\boxed{\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}}$$

Normalized Beam Emittance

- Hence, as particles are accelerated, the area in *x-x'* phase space is not preserved, while area in x-p_x is preserved. Thus, we define a "normalized" beam emittance, as

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{\text{Lorentz}}$$

- In principle, the *normalized* beam emittance should be preserved during acceleration, and hence along the chain of accelerators from source to target. Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.
- In practice, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors and setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep the emittance as small as possible.

Let's Think About the Numbers & Units…

$$\epsilon_N = (\beta\gamma)\epsilon \qquad\qquad \epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If <x^2> ~ mm^2, and <x'^2> ~ mrad^2, then the emittance can have units of mm-mrad (*also* = μm)
- Courant-Snyder parameters (or, *Twiss parameters*)

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm^2/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad^2/(mm-mrad) ~ 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- So, can look at propagation of amplitude function through beam line given matrices of individual elements. Beam size at a particular location determined by

$$x_{rms}(s) = \sqrt{\beta(s)\epsilon_N/\pi(\beta\gamma)}$$

- Or, given an initial particle distribution, can look at propagation of second moments (of position, angle) given the same element matrices, and hence the propagation of the beam size, $\sqrt{\langle x^2\rangle(s)}$.
- Note: so far, have neglected:
  - dispersion of trajectories due to momentum (*coming up*)
  - hor-ver coupling (typically zero by design, but not always)
  - intensity dependent effects (*e.g.*, space charge)
  - and probably a few other details for certain conditions...

Tomorrow

- Lecture 1
  - Overview of types and uses of accelerators
  - Single-pass vs. repetitive systems
  - Transverse vs. longitudinal motion
  - Beams and particle distributions
  - Transverse beam optics

- Lecture 2
  - Dispersion
  - Longitudinal beam dynamics
    - »bunchers, re-bunchers; buckets and bunches
  - Optics modules
  - Accelerators for nuclear and high energy physics
  - Future directions