syphers@msu.edu

Some Fundamentals of Modern Particle Accelerators

Mike Syphers

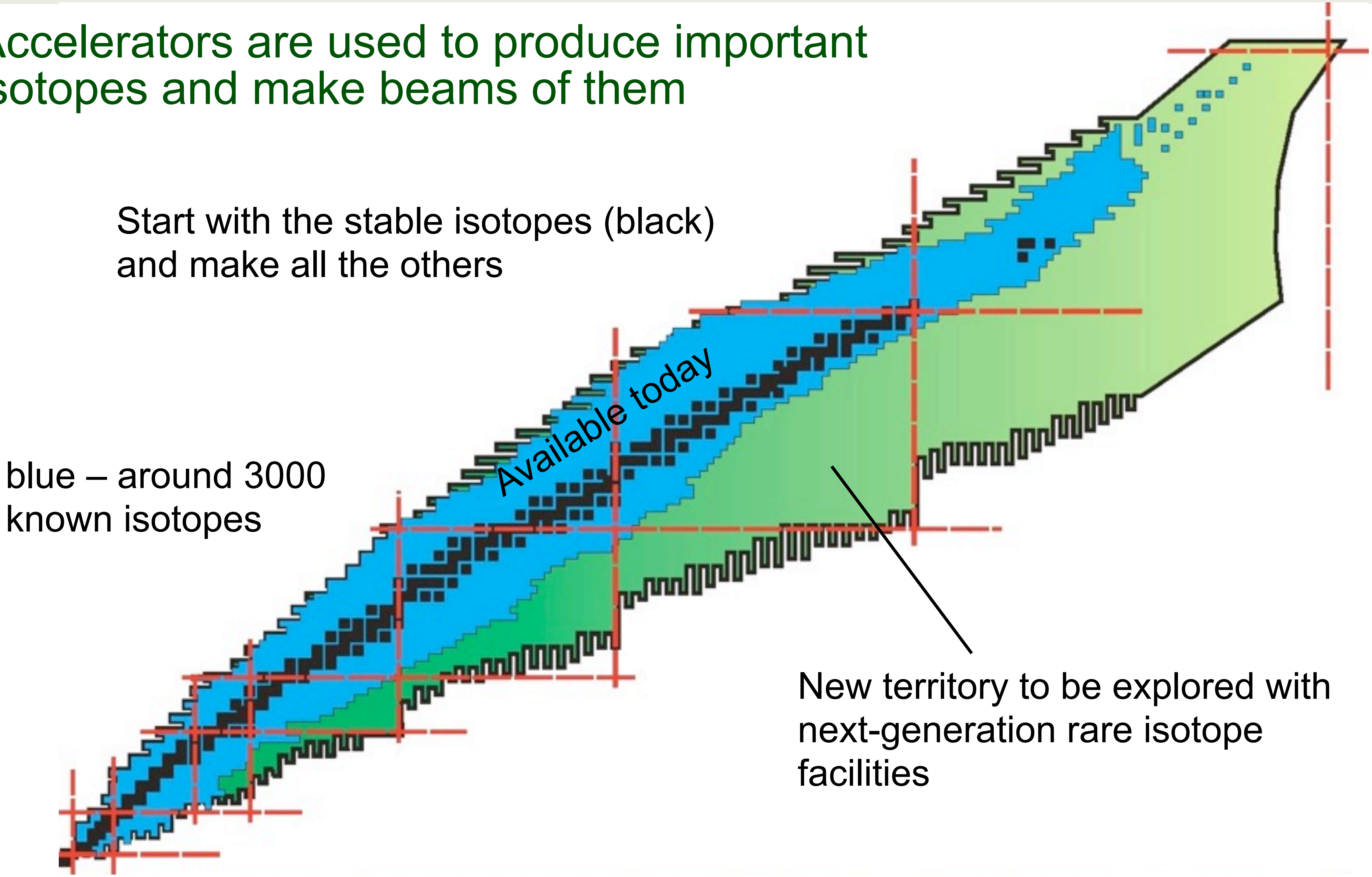
Michigan State University

National Superconducting Cyclotron Laboratory

Facility for Rare Isotope Beams

Production of Isotope Beams

- Accelerators are used to produce important isotopes and make beams of them



Production of Rare Isotope Beams

- There is a variety of nuclear reaction mechanisms used to add or remove nucleons:
  - Spallation, Fragmentation, Coulomb fission (photo fission), Nuclear induced fission, Light ion transfer, Fusion-evaporation (cold, hot, incomplete, …), Fusion-Fission, Deep Inelastic Transfer, Charge Exchange, ...
- The accelerator system produces a primary beam of charged particles and delivers them to a target
  - e.g., use protons for spallation, heavy ions for fragmentation, *etc.*
- *Thus,* once created in a source, need to accelerate ions, direct them along a desired trajectory, and keep them contained along the way

accelerating devices

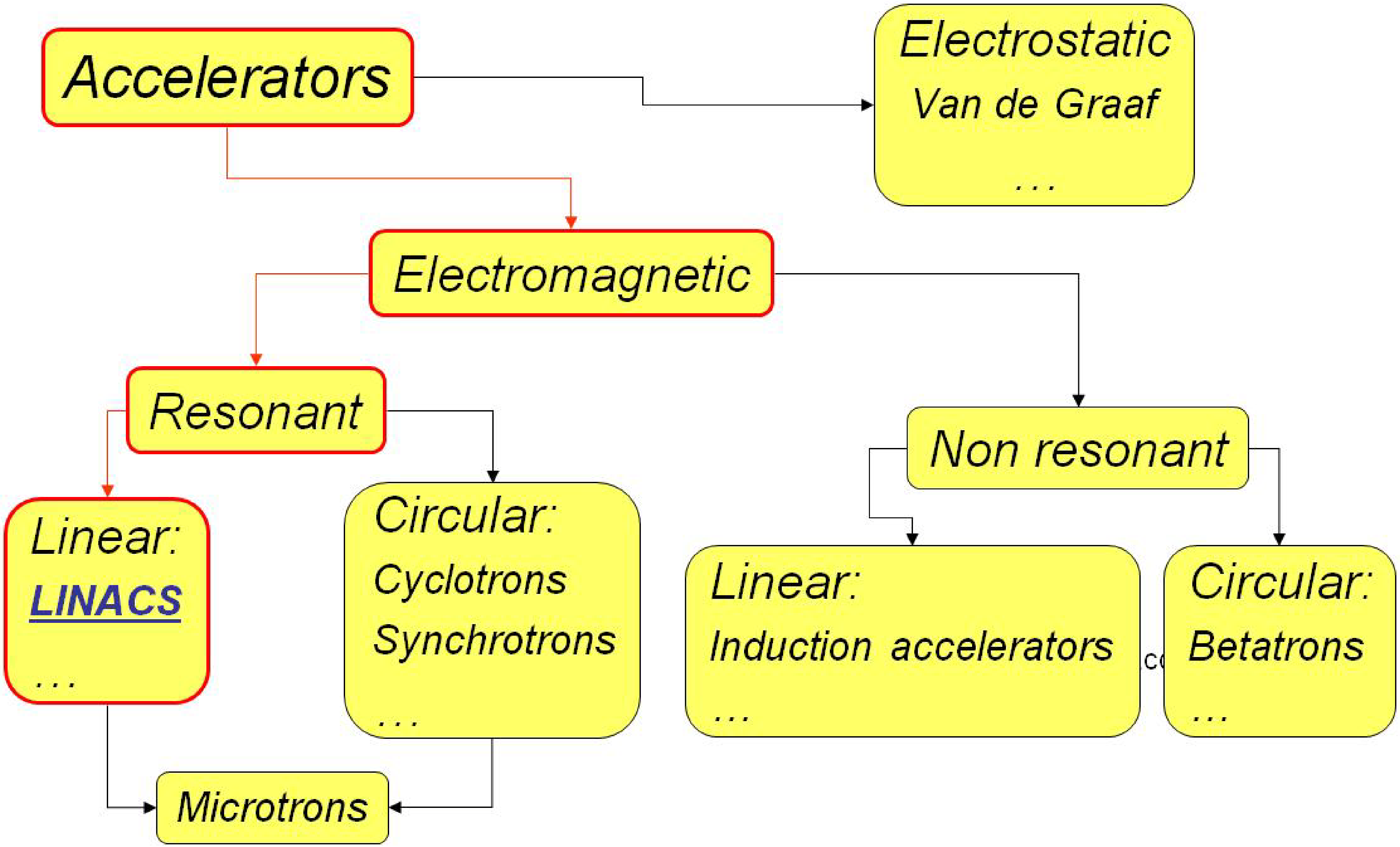
steering devices

focusing devices

Accelerators

- The particle accelerator used for production is often called the "driver"
- Various types...
  - Cyclotron
    - » NSCL, GANIL, TRIUMF (proton driver), HRIBF (proton driver), RIKEN RIBF
  - Synchroton
    - » GSI, FAIR-GSI
  - Linear Accelerator (*LINAC*): ATLAS (ANL), FRIB (MSU)
  - Others like Fixed-Field Alternating Gradient accelerators currently not used
- Main Parameters
  - Top Energy (*e.g*. FRIB will have 200 MeV/u uranium ions)
  - Particle range (TRIUMF cyclotron accelerates H, hence is used for spallation
  - Intensity or Beam Power
    - » beam intensity: # particles /sec = $dN/dt = I/Qe$ 1 pμA == 6×10^{12} /s
    - » *Power* = dN/dt [pμA] x *Particle Energy* [GeV] *e.g.*, 400 kW = 8 [pμA] x 50 [GeV]
    - » If particle of mass Am_u and charge Qe has energy E, then can write
      - $Power$ [W] $= dN/dt \times E = dN/dt \times (A \times E/A)$ == $(A/Q) \times I$ [μA] $\times E/A$ [MeV/u]/e
      - etc., ...

Accelerator Families



A Little Accelerator History

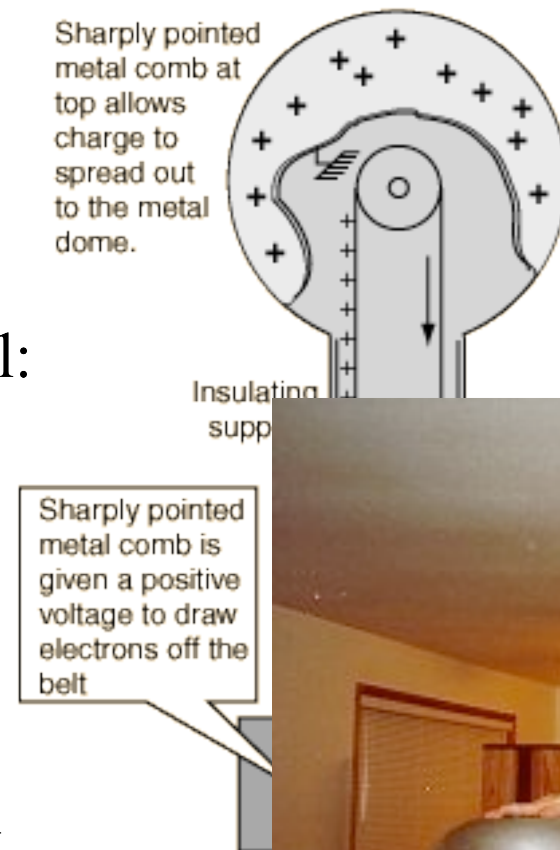
▪DC Acceleration

1927: Lord Rutherford requested a "copious supply" of projectiles more energetic than natural alpha and beta particles. At the opening of the resulting High Tension Laboratory, Rutherford went on to reiterate the goal:

"What we require is an apparatus to give us a potential of the order of 10 million volts which can be safely accommodated in a reasonably sized room and operated by a few kilowatts of power. We require too an exhausted tube capable of withstanding this voltage… I see no reason why such a requirement cannot be made practical."

Van de Graaff

(1929)



A Little Accelerator History

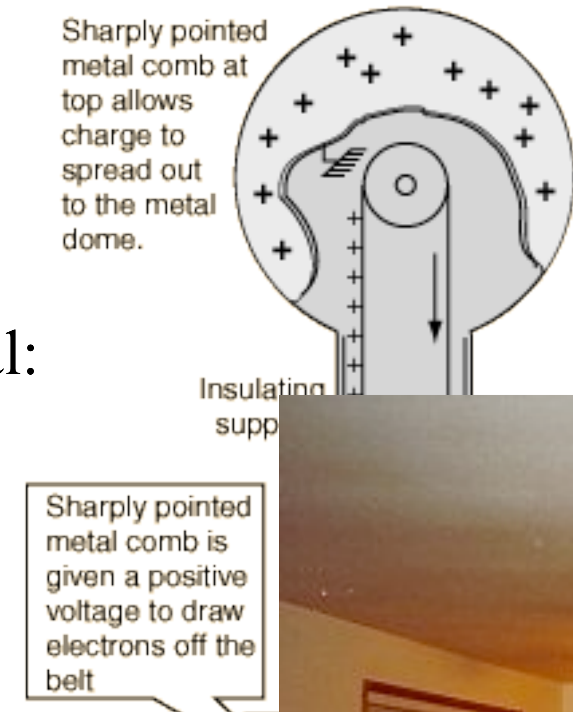
▪DC Acceleration

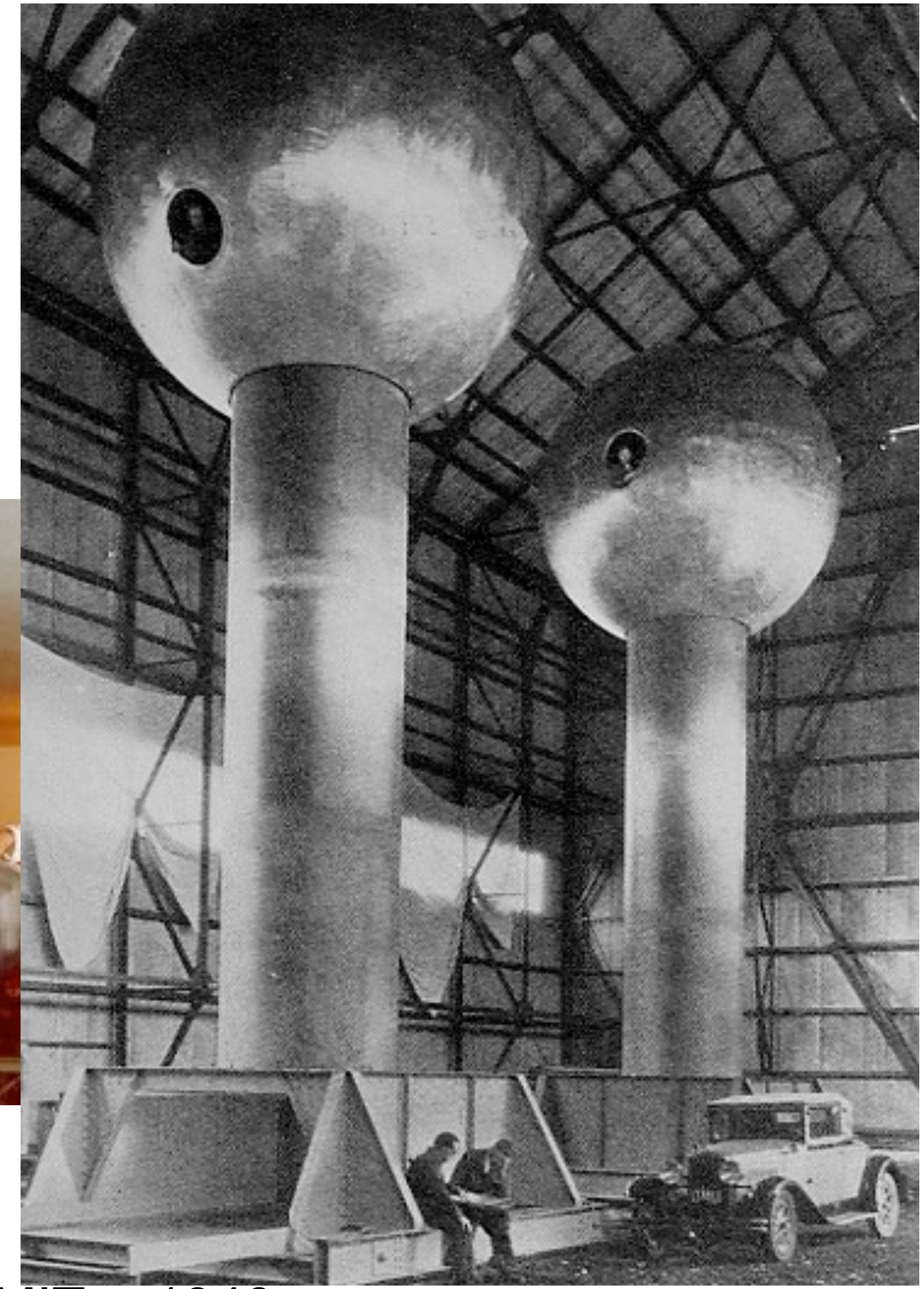
Van de Graaff

(1929)

1927: Lord Rutherford requested a "copious supply" of projectiles more energetic than natural alpha and beta particles. At the opening of the resulting High Tension Laboratory, Rutherford went on to reiterate the goal:

"What we require is an apparatus to give us a potential of the order of 10 million volts which can be safely accommodated in a reasonably sized room and operated by a few kilowatts of power. We require too an exhausted tube capable of withstanding this voltage… I see no reason why such a requirement cannot be made practical."





MIT, c.1940s

A Little Accelerator History

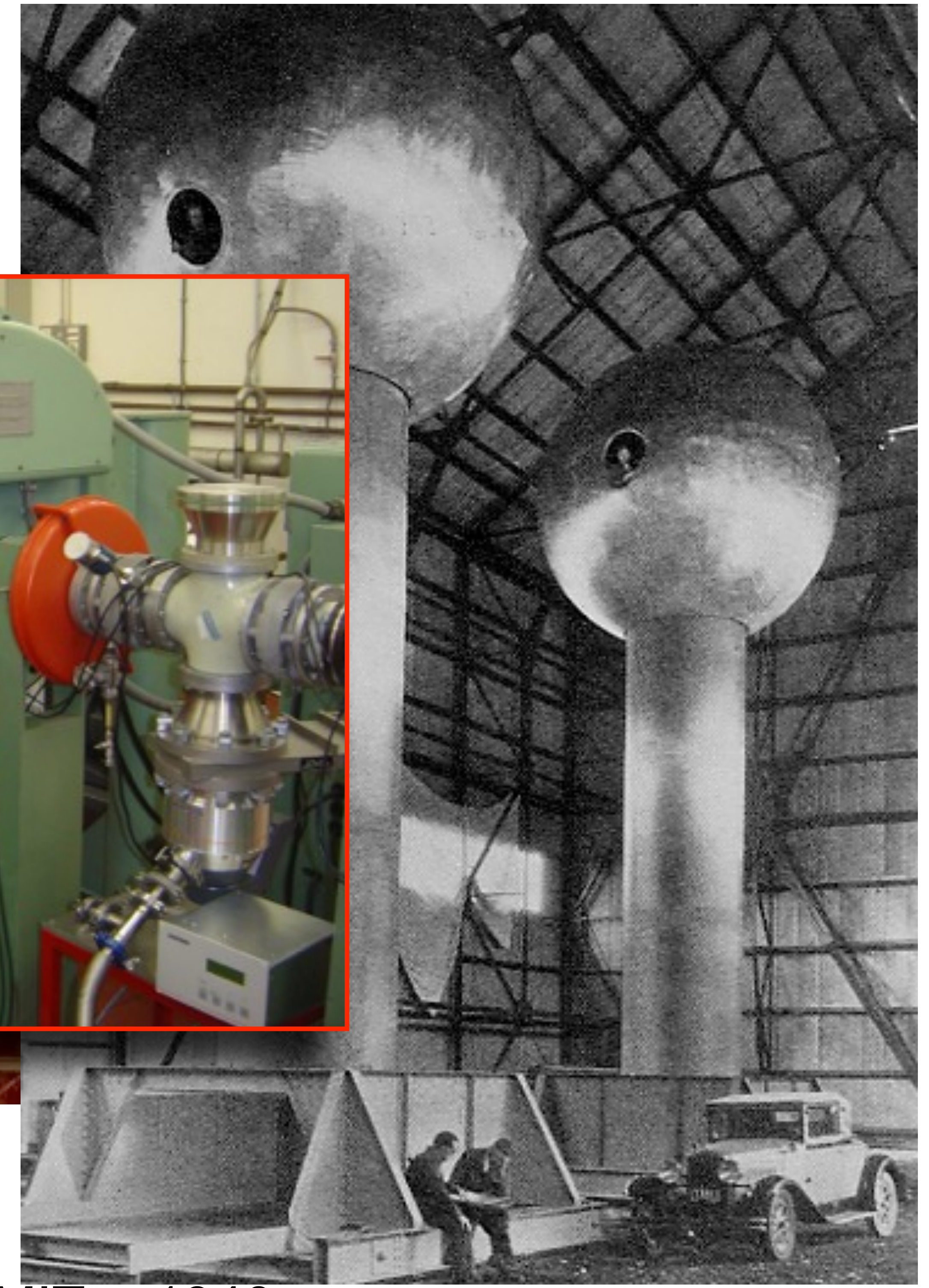
▪DC Acceleration

1927: Lord Rutherford requested a "copious supply" of projectiles more energetic than natural alpha a
particles. At the opening of t
resulting High Tension Labor
Rutherford went on to reitera

"What we require is an app
give us a potential of the orde
million volts which can be sa
accommodated in a reasonab
room and operated by a few k
of power. We require too an
tube capable of withstanding
voltage… I see no reason wh
requirement cannot be made practical."

Van de Graaff

(1929)



MIT, c.1940s

Cockcroft and Walton

- Voltage Multiplier

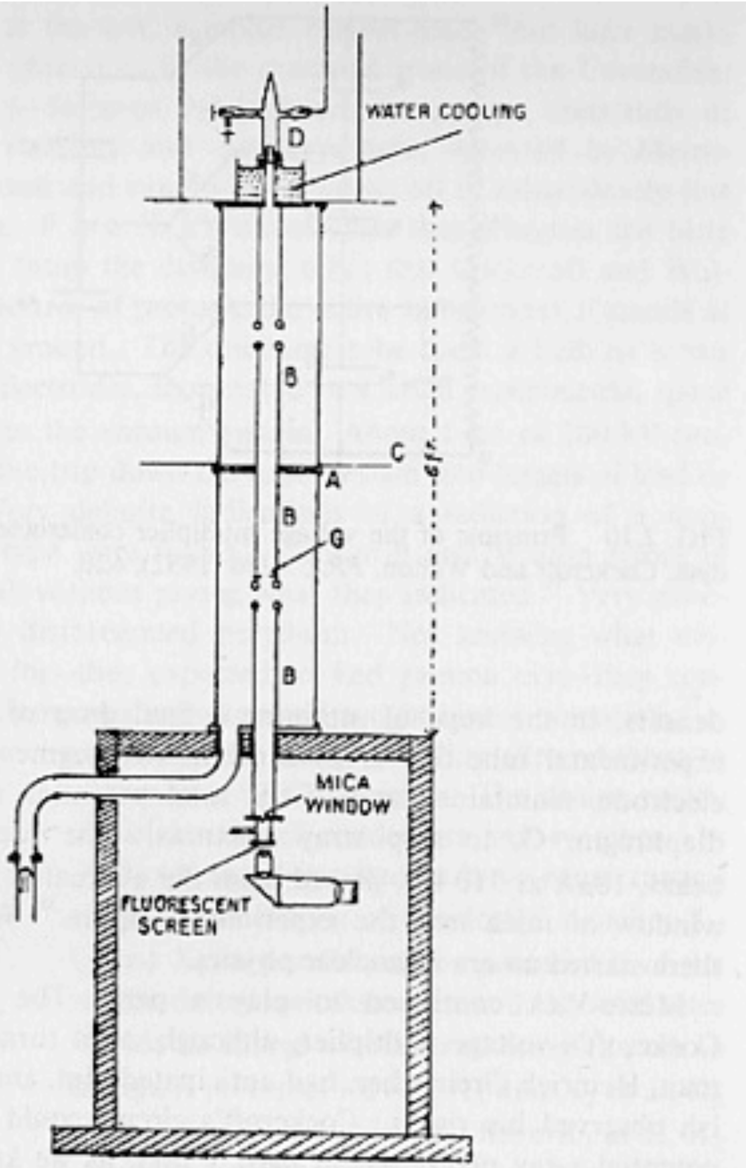
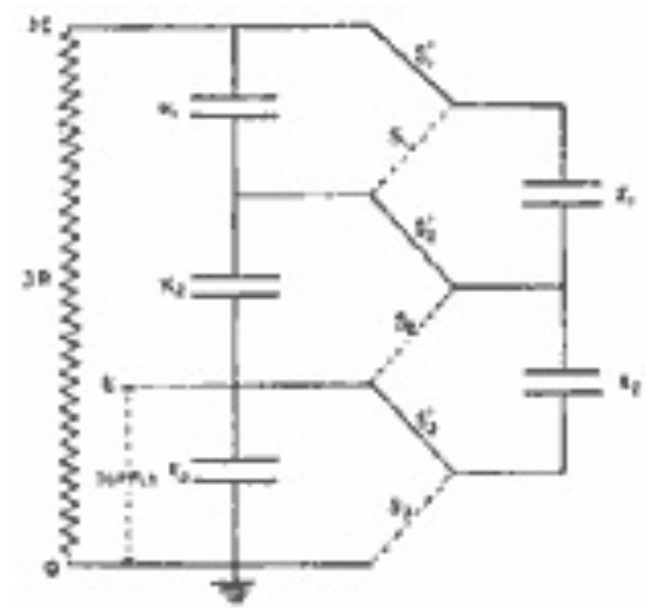
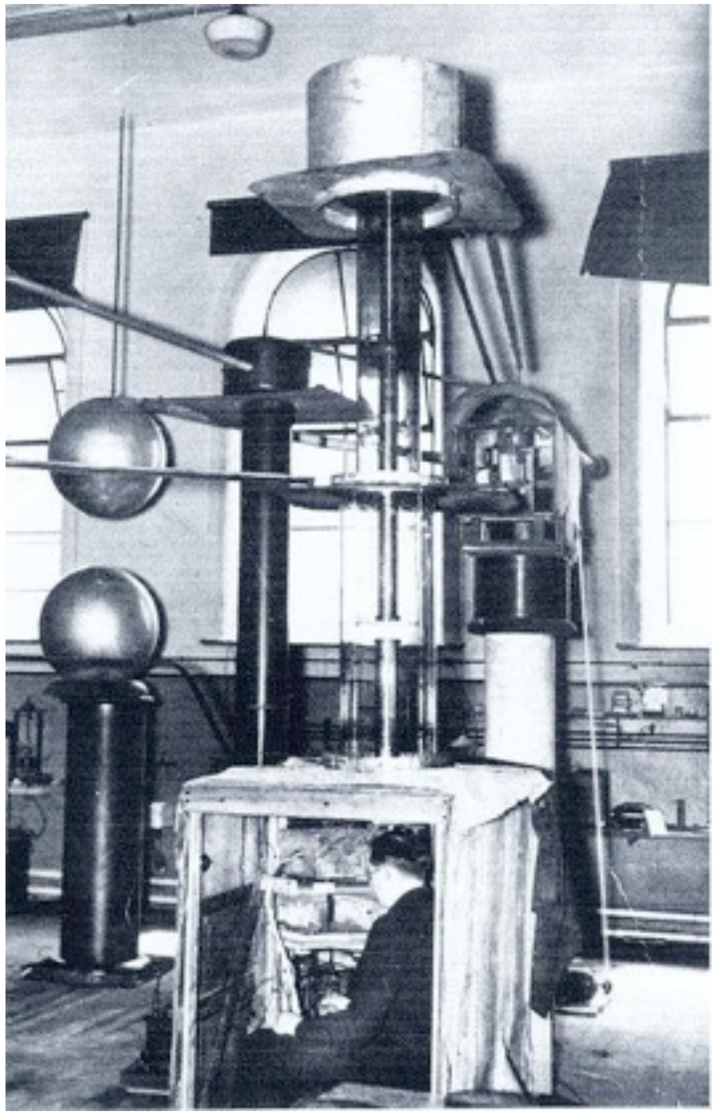


FIG. 2.11 Accelerating tube and target arrangement of the Cockcroft-Walton machine. The source is at D; C is a metallic ring joint between the two sections of the constantly pumped tube. The mica window closes the evacuated space. Cockcroft and Walton, *PRS, A136* (1932), 626.



Converts AC voltage V to DC voltage n x V



Cockcroft and Walton

- Voltage Multiplier

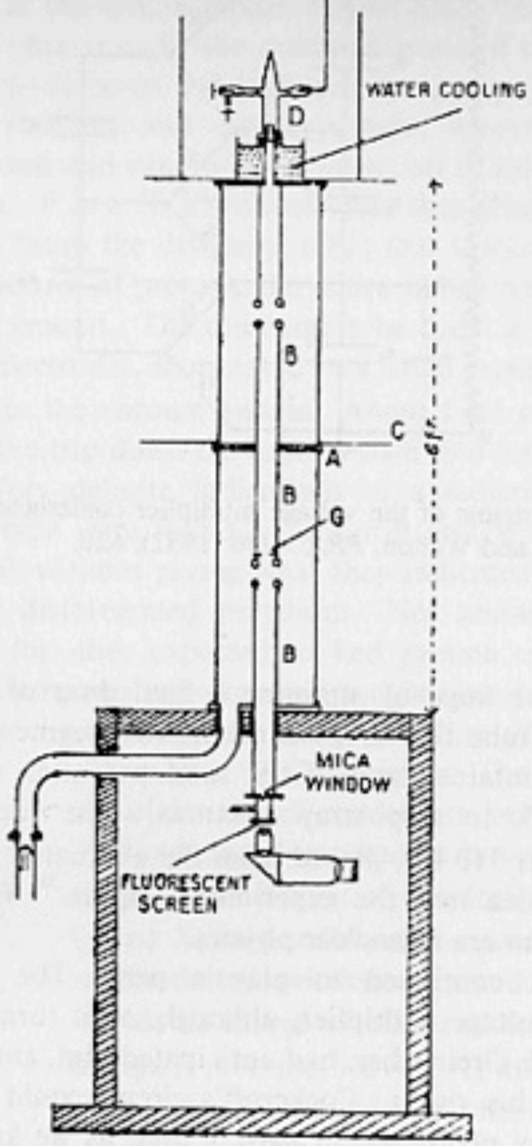
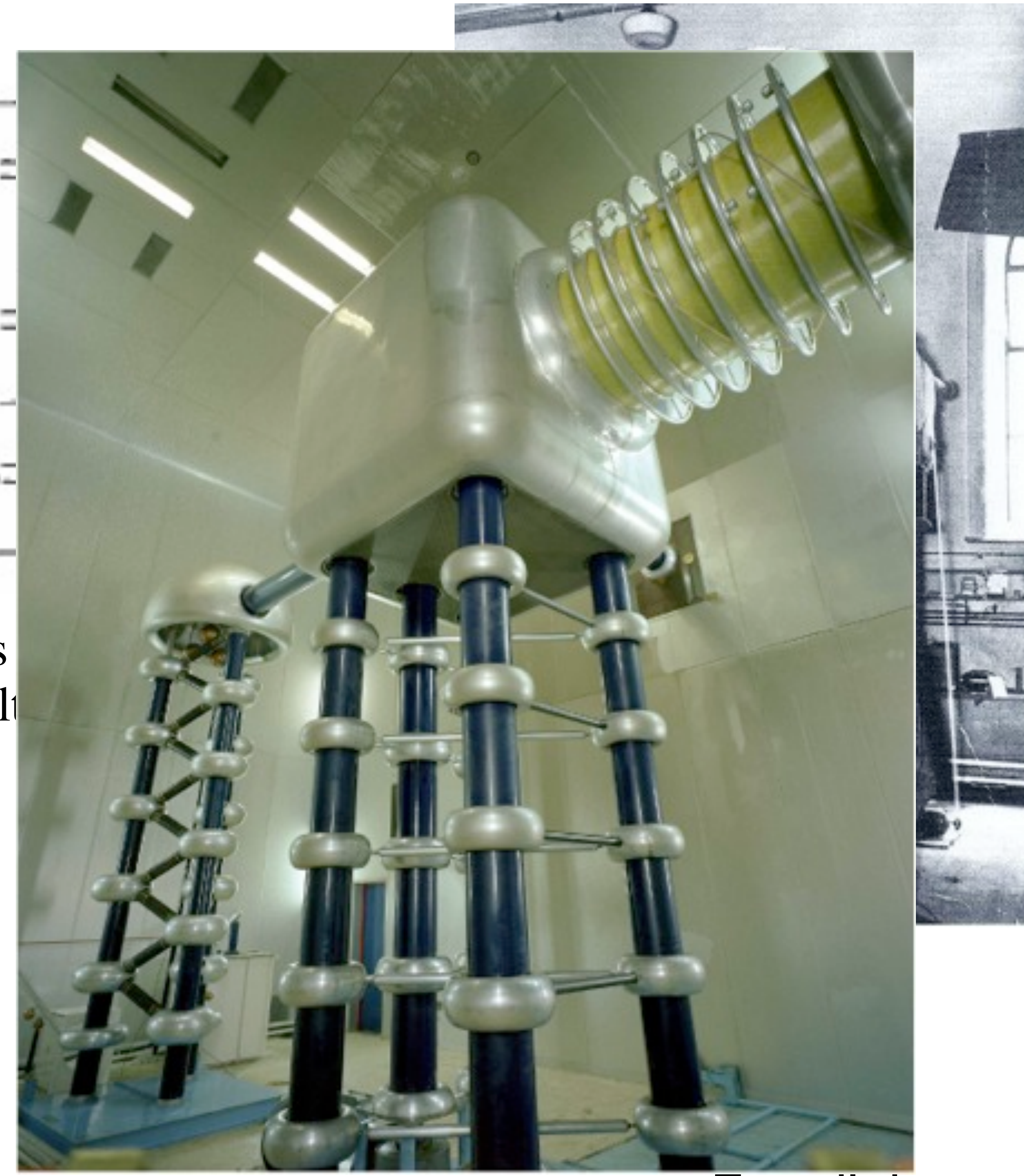


FIG. 2.11 Accelerating tube and target arrangement of the Cockcroft-Walton machine. The source is at D; C is a metallic ring joint between the two sections of the constantly pumped tube. The mica window closes the evacuated space. Cockcroft and Walton, *PRS, A136* (1932), 626.

Converts
DC volt

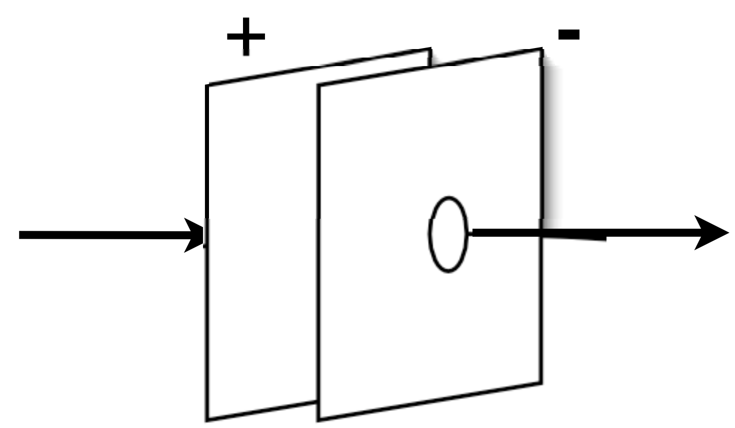


Fermilab

The Route to Higher Energies

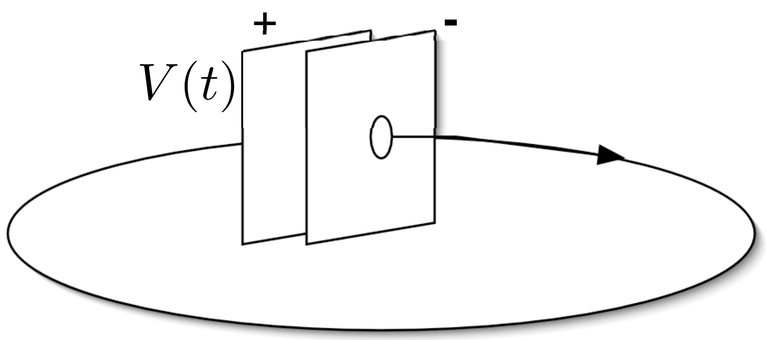
- The Need for AC Systems

$$\text{energy gain} = q \cdot V$$

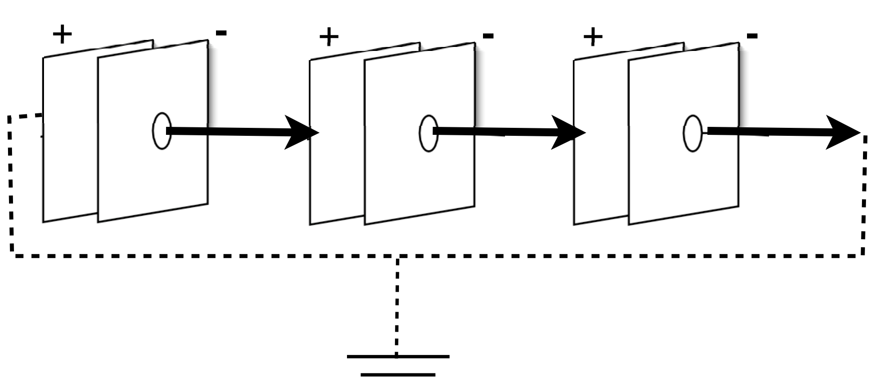


DC systems limited to a few MV

Circular Accelerator



Linear Accelerator



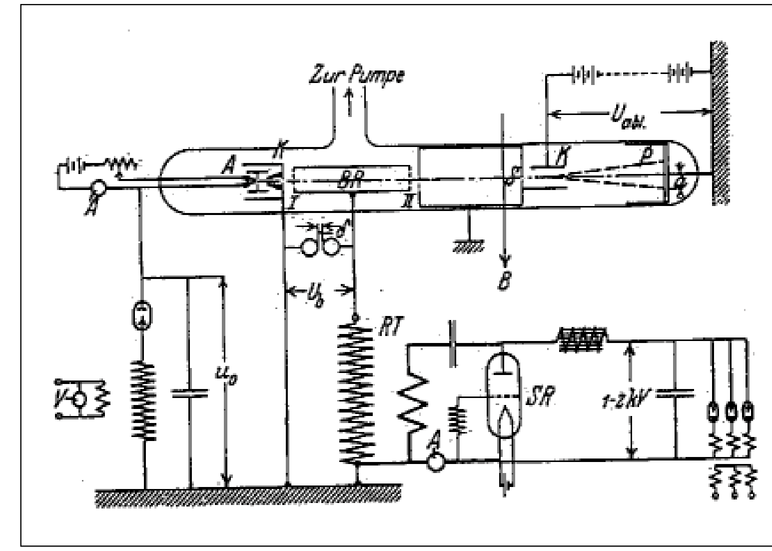
$$\oint (q\vec{E}) \cdot d\vec{s} = work = \Delta(energy)$$

To gain energy, a time-varying field is required:

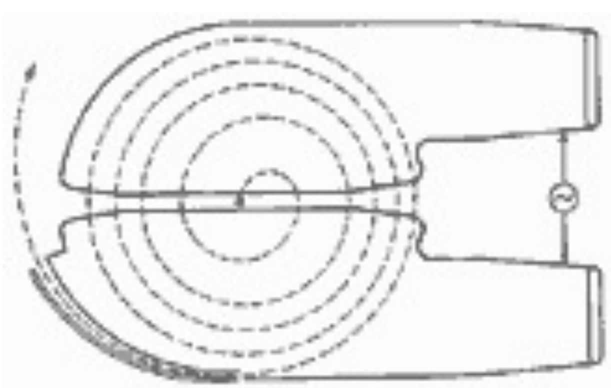
$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$

The Linear Accelerator

- The linear accelerator (linac) -- 1928-29
  - Wideroe (U. Aachen; grad student!)
    - » Dreamt up concept of “Ray Transformer” (later, called the “Betatron”); thesis advisor said was “sure to fail,” and was rejected as a PhD project. Not deterred, illustrated the principle with a “linear” device, which he made to work -- got his PhD in engineering
  - 50 keV; accelerated heavy ions (K+, Na+)
  - utilized oscillating voltage of 25 kV @ 1 MHz



- The Cyclotron -- 1930's, Lawrence (U. California)
  - read Wideroe’s paper (actually, looked at the pictures!)
  - an extended “linac” unappealing -- make it more compact:

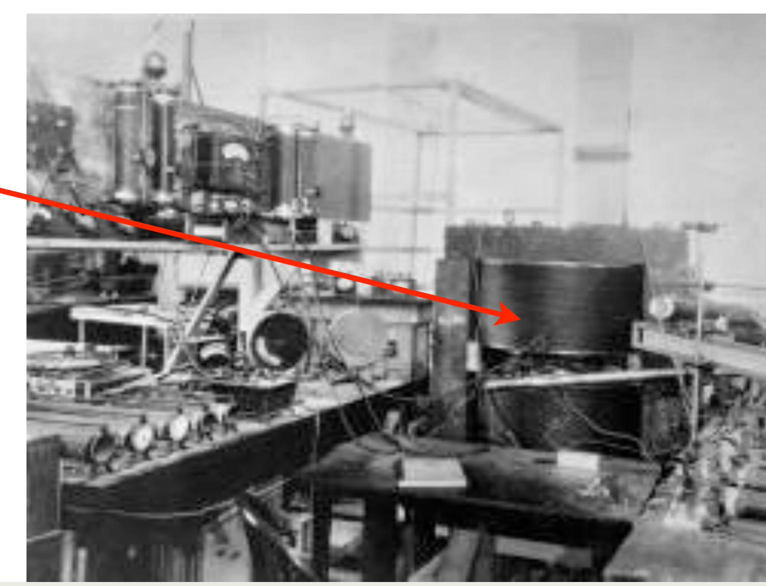


V

$$\frac{1}{T} = \frac{q \cdot B}{2\pi m}$$

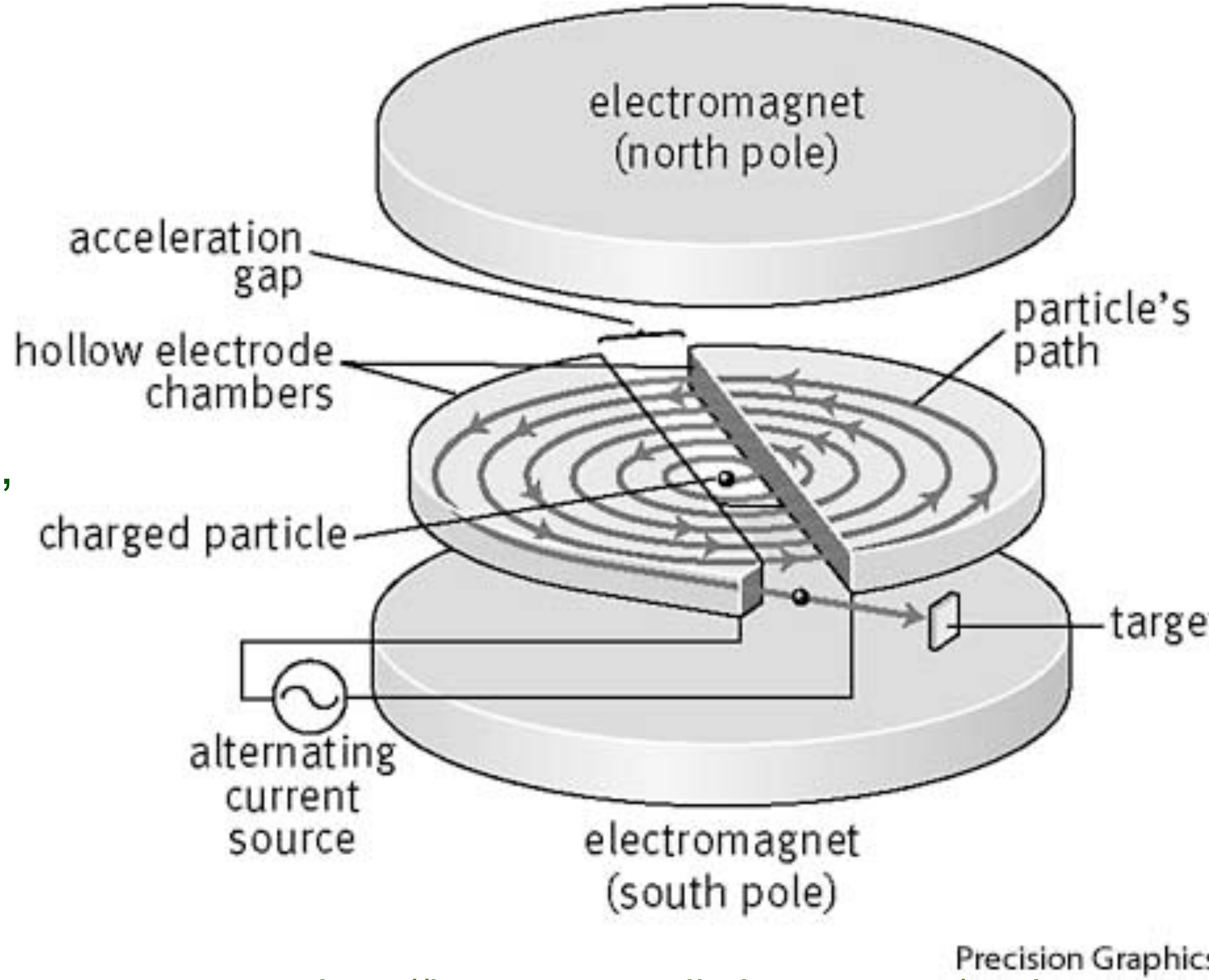
4.5 inch diameter!

11 inch diameter



Cyclotrons

- Relatively easy to operate and tune (only a few parts).
- Tend to be used for isotope production and places where reliable and reproducible operation are important
- Intensity is moderately high, acceleration efficiency is high, cost low
- Relativity is an issue, so energy is limited to a few hundred MeV/u.
- RIKEN Superconducting Ring Cyclotron 350 MeV/u



http://images.yourdictionary.com/cyclotron

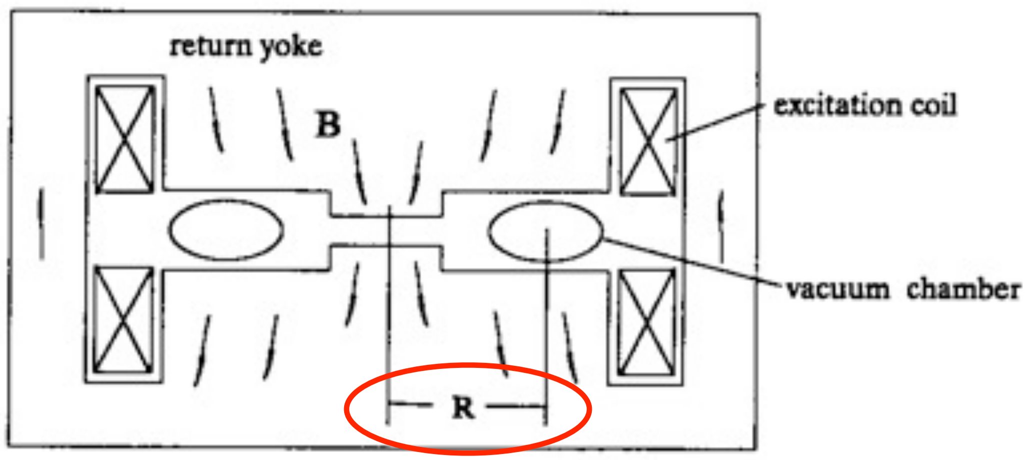
60-inch Cyclotron, Berkeley -- 1930's

184-inch Cyclotron, Berkeley -- 1940's

Meeting up with Relativity

- The Synchrocyclotron (FM cyclotron) -- 1940's
  - beams became relativistic (esp. e^-) --> oscillation frequency no longer independent of momentum; cyclotron condition no longer held throughout process; thus, modulate freq.

- The Betatron -- 1940, Kerst (U. Illinois)
  - induction accelerator
    - » $$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$
    - »used for electrons
    - »Beam dynamics heavily studied
      - “betatron oscillations”



field on orbit of radius *R*

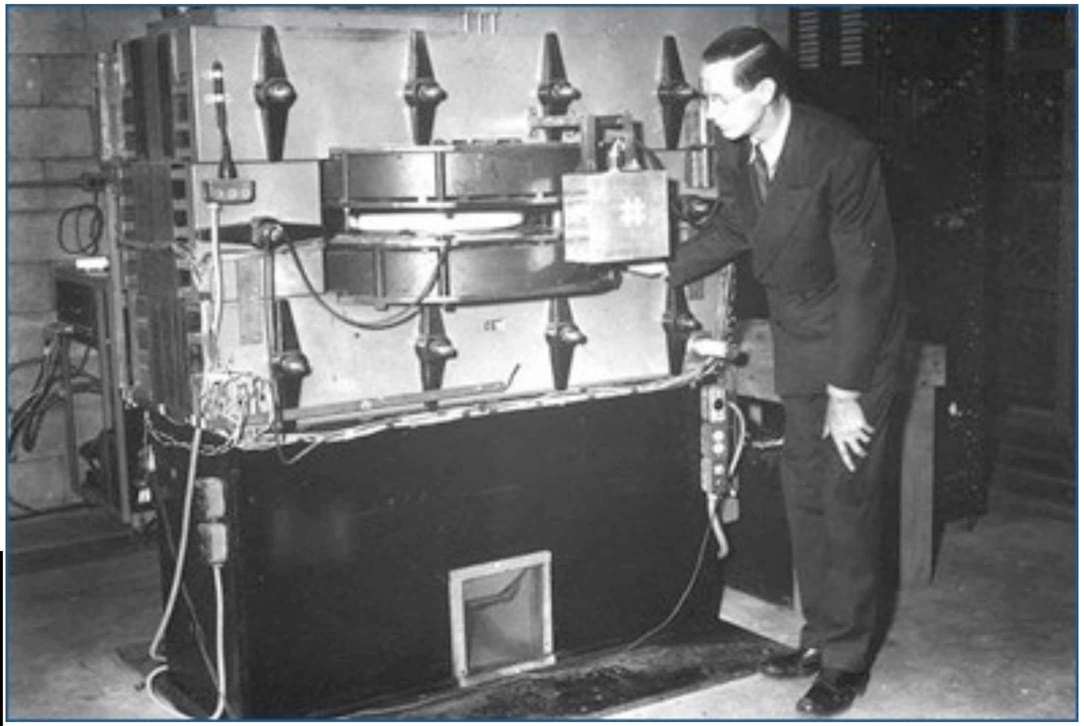
$$\boxed{\frac{d\Phi}{dt} = 2(\pi R^2)\frac{dB_z}{dt}}$$

Meeting up with Relativity

- The Synchrocyclotron (FM cyclotron) -- 1940's
  - beams became relativistic (esp. e^-) --> oscillation frequency no longer independent of momentum; cyclotron condition no longer held throughout process; thus, modulate freq.
- The Betatron -- 1940, Kerst (U. Illinois)
  - induction accelerator
    - » $$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$
    - »used for electrons
    - »Beam dynamics heavily studied
      - "betatron oscillations"

field on orbit of radius *R*

$$\boxed{\frac{d\Phi}{dt} = 2(\pi R^2)\frac{dB_z}{dt}}$$

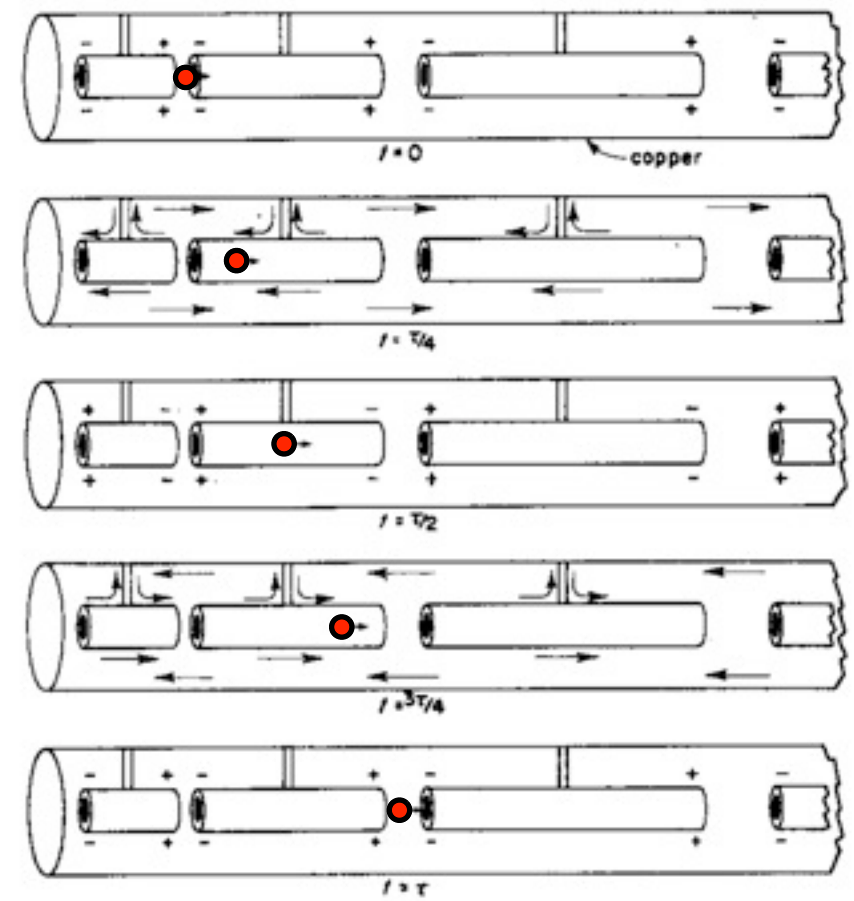


~ 2 MeV; later models --> 300 MeV

- The Microtron --1944, Veksler (Russia)
  - use one cavity with one frequency, but vary path length each "revolution" as function of particle speed

The “Modern” Linear Accelerator

- Alvarez -- 1946 (U. California)
  - cylindrical cavity with drift tubes
  - particles “shielded” as fields change sign
  - most practical for protons, ions
  - GI surplus equip. from WWII Radar technology

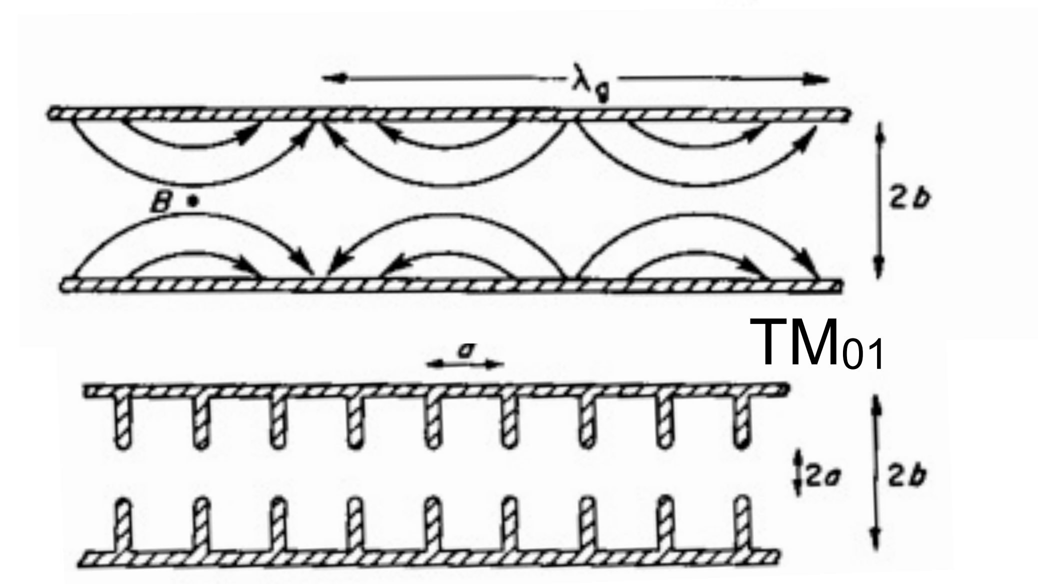


- Traveling-Wave Electron Accelerator --

  c.1950 (Stanford, + Europe)

  - TM_{01} waveguide arrangement

  - iris-loaded cylindrical waveguide
    - »match phase velocity w/ particle velocity...

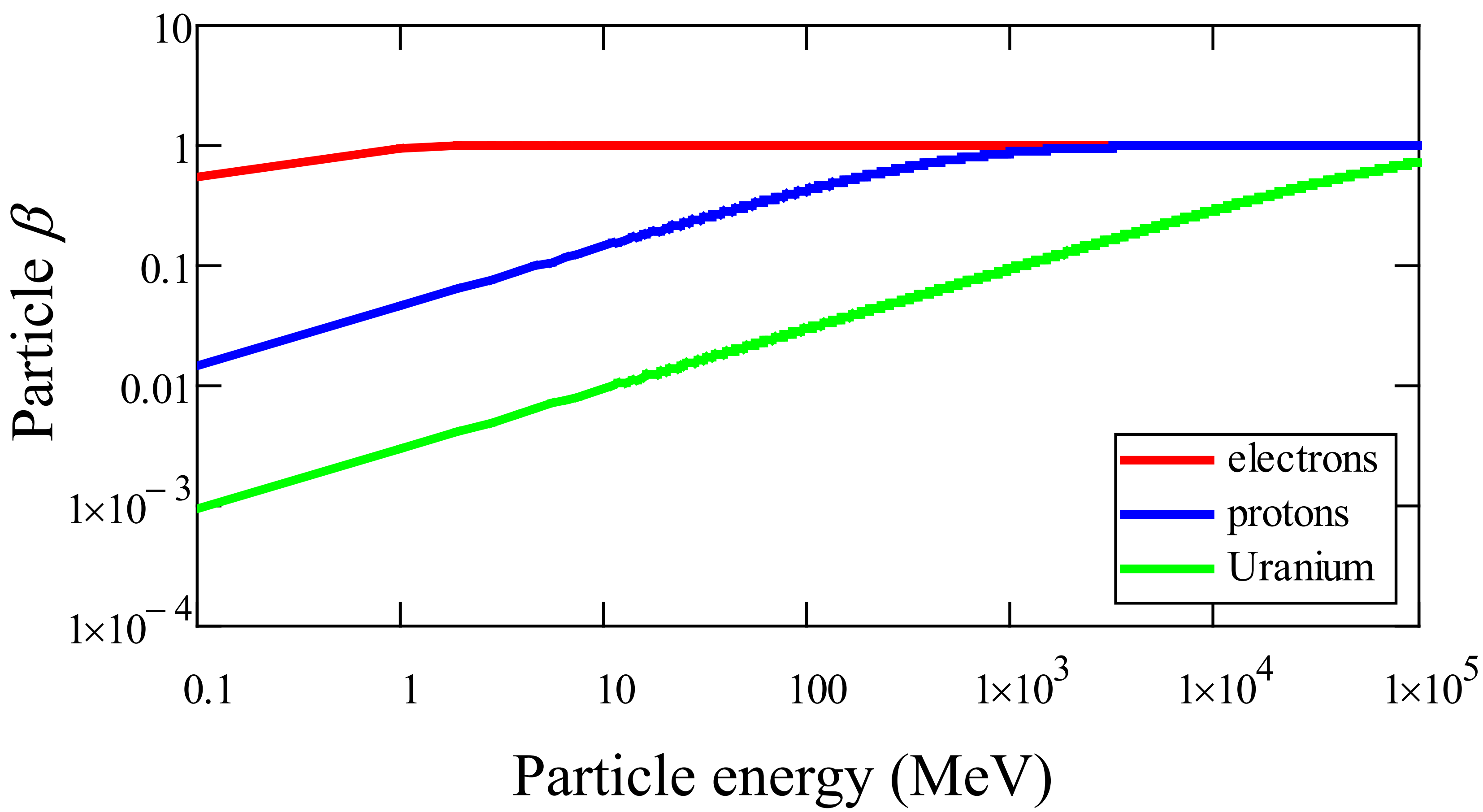


Different Arrangements for Different Particles

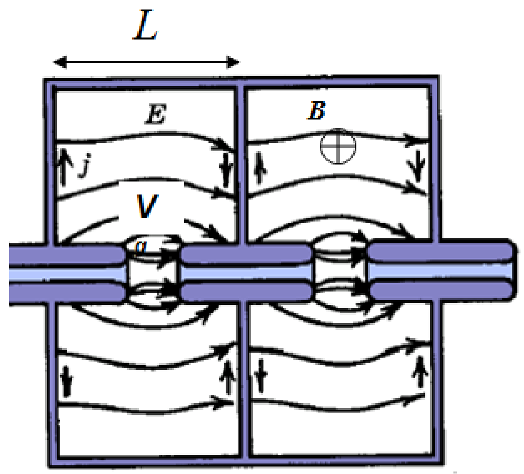
- Accelerating system used will depend upon the evolution of the particle velocity along the system
  - electrons reach a constant velocity at relatively low energy
    - » thus, can use one type of resonator
  - heavy particles reach a constant velocity only at very high energy
    - » thus, may need different types of resonators, optimized for different velocities

Particles rest mass:

- *e* *0.511 MeV*
- *p* *938 MeV*
- *^{239}U* *~220000 MeV*

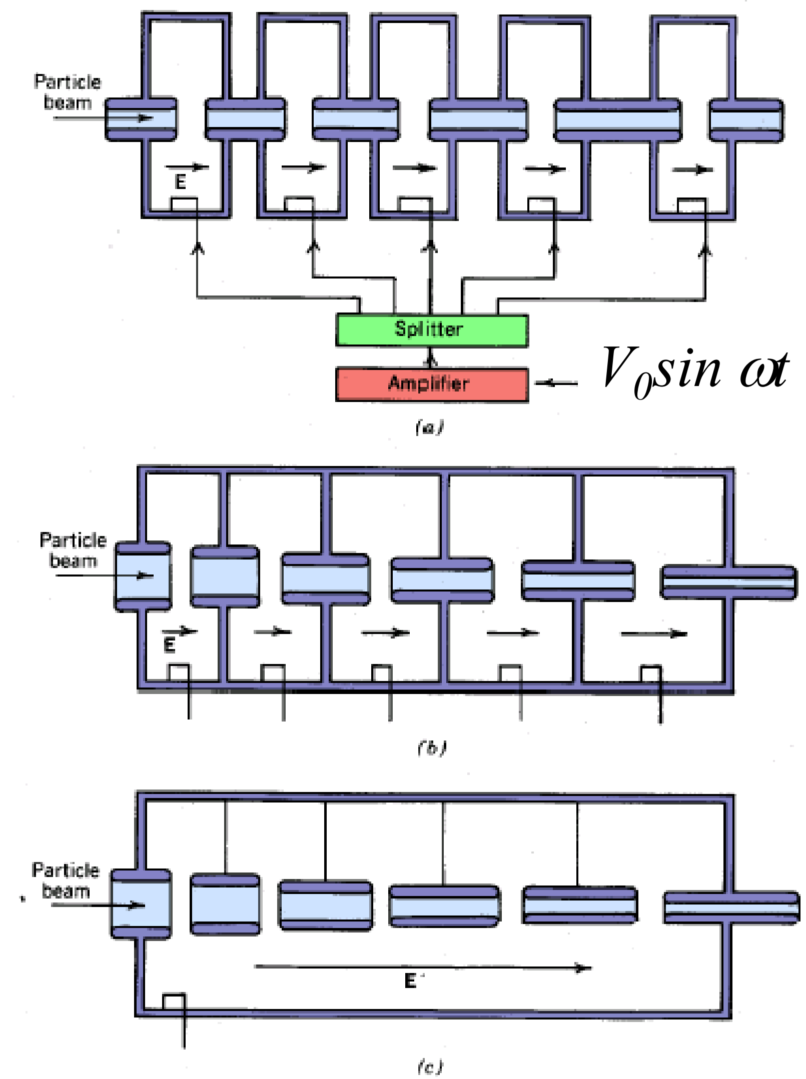


Radiofrequency Resonant Cavities



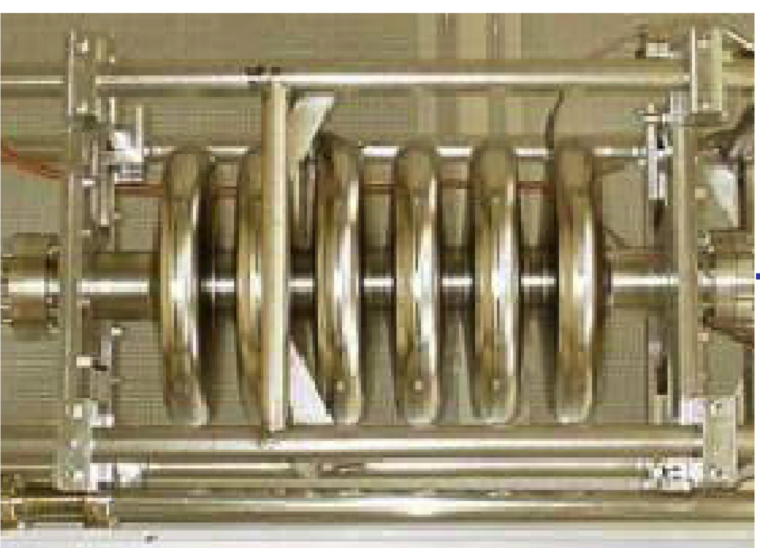
$$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt}$$

Time varying: we can use many cavities in series!



- *Resonant cavities reduce rf power consumption, increase gradient and efficiency*
- *Long cavities (with many gaps) are generally more efficient*

Accelerating field	$E_a = V_g/L$
Stored EM energy	$U \propto E_a^2$
Quality Factor	$Q = \omega U/P = \Gamma/R_s$



A. Facco –FRIB and INFN *SRF Low-beta Accelerating Cavities for FRIB* *MSU 4/10/2011*

Normal vs. Superconducting Cavities

DTL tank - Fermilab

Normal conducting

Cu cavity @ 300K

$R_s \sim 10^{-3}\ \Omega$

$Q \sim 10^4$

Superconducting

Nb Cavity @ 4.2K

$R_s \sim 10^{-8}\ \Omega$

$Q \sim 10^9$

LNL PIAVE 80 MHz, β =0.047 QWR

Superconductivity allows

- *great reduction of rf power consumption even considering cryogenics (1W at 4.2K ~ 300W at 300K)*
- *the use of short cavities with wide velocity acceptance*

Low-β Superconducting Cavities

- Can use regularly spaced cavities when particle velocity is not changing much -- i.e., when $v \sim c$

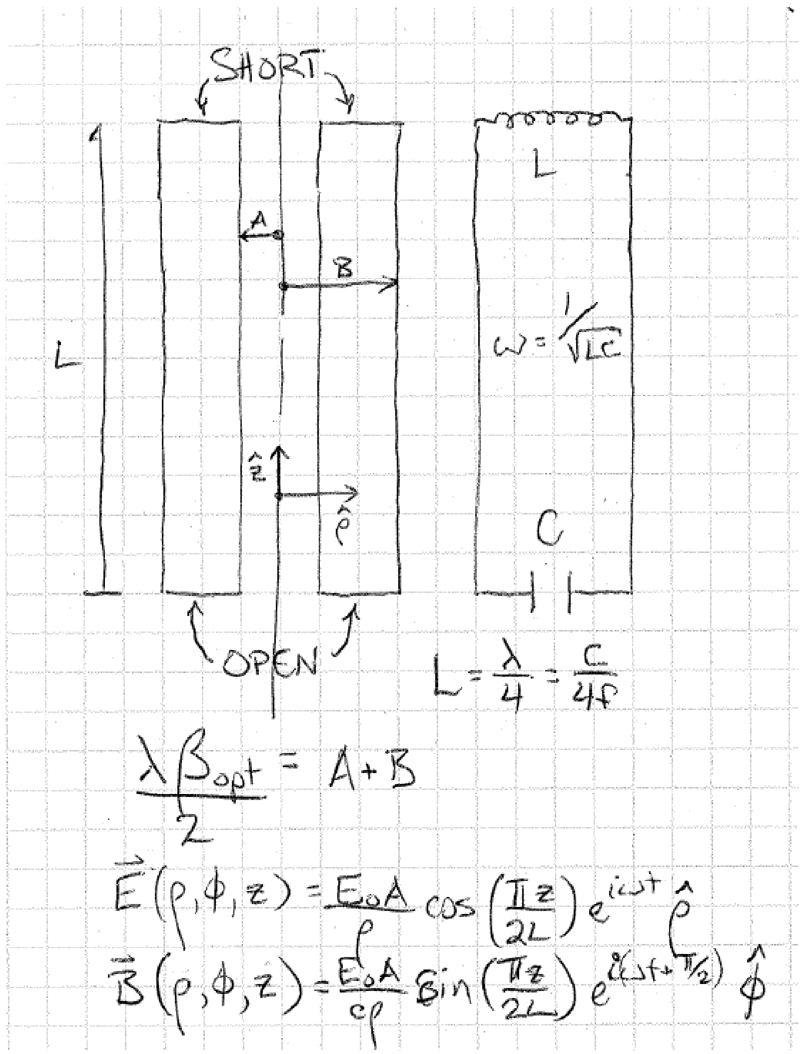
- For "slow" particles, in which velocity changes are dramatic between accelerating gaps, various solutions/designs...

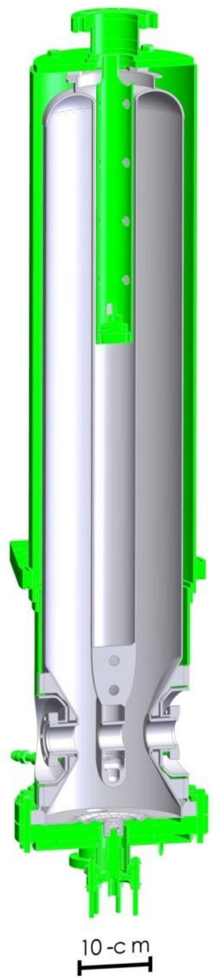
A. Facco

β<1 resonators, from very low (β~0.03) to intermediate (β~0.5): many different shapes and sizes

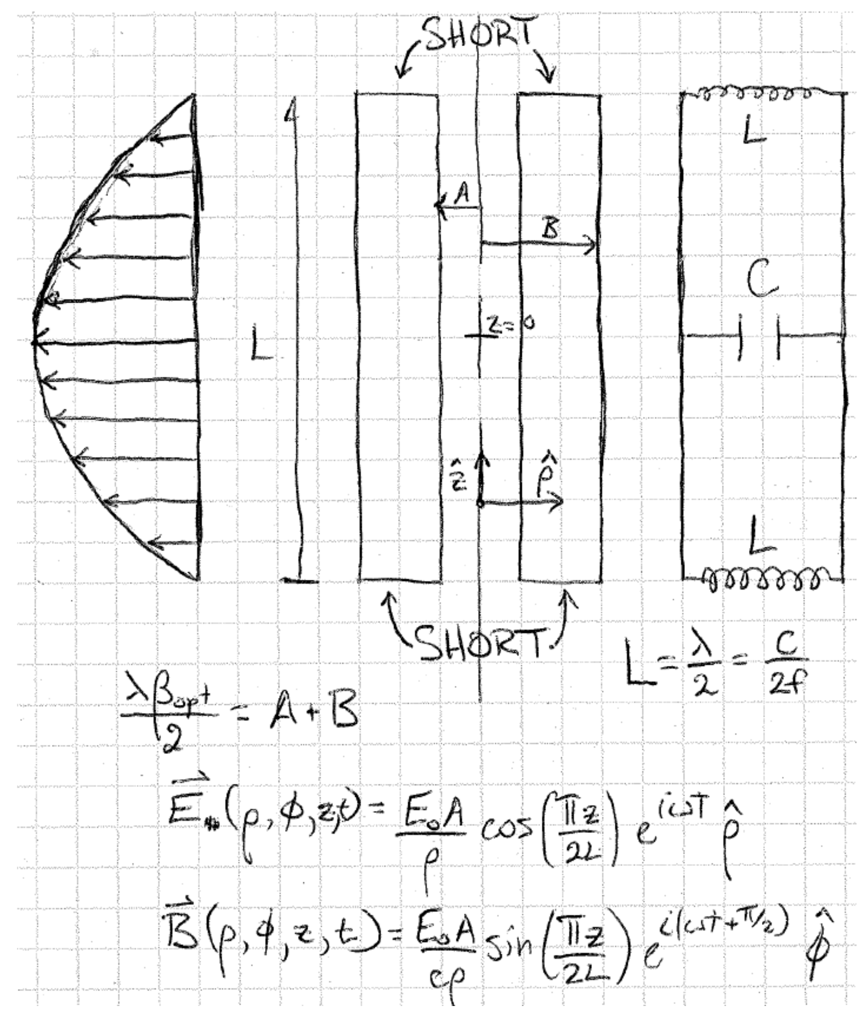
λ/4-, λ/2-Wave Resonators

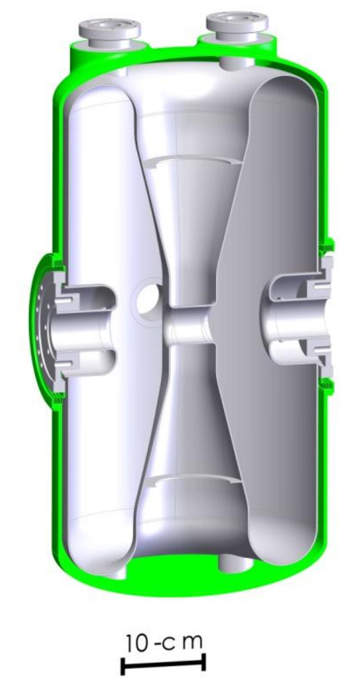
- Quarter-Wave Resonator:





- Half-Wave Resonator:



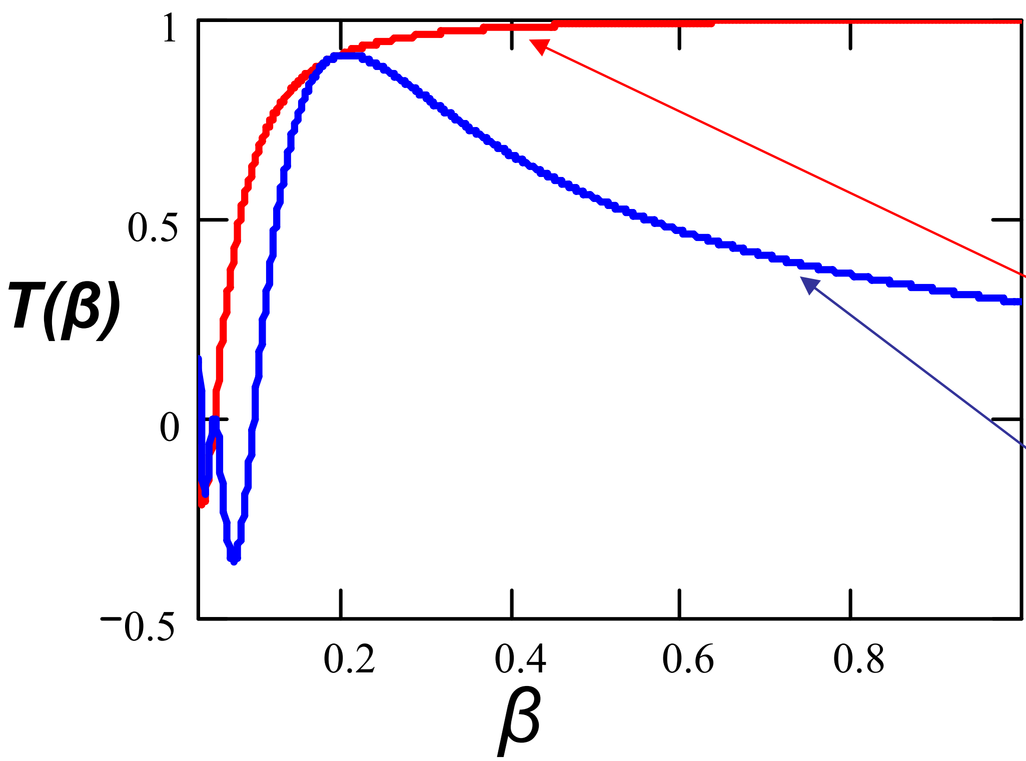


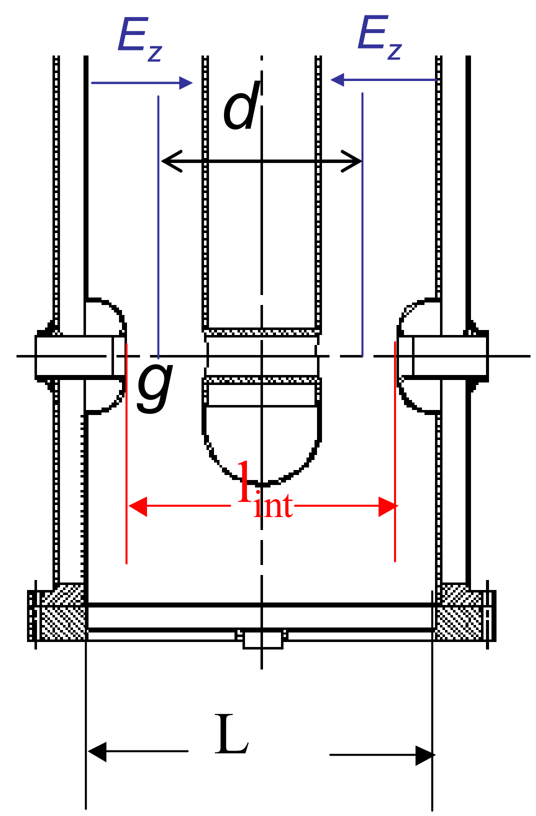
J. Holzbauer

Transit Time Factor for 2-gap π-mode Cavity

(constant E_z approximation)

$$T(\beta) \cong \frac{\sin\left(\frac{\pi g}{\beta\lambda}\right)}{\left(\frac{\pi g}{\beta\lambda}\right)} \sin\left(\frac{\pi d}{\beta\lambda}\right)$$





1° term: 1-gap effect → *g<βλ/2*

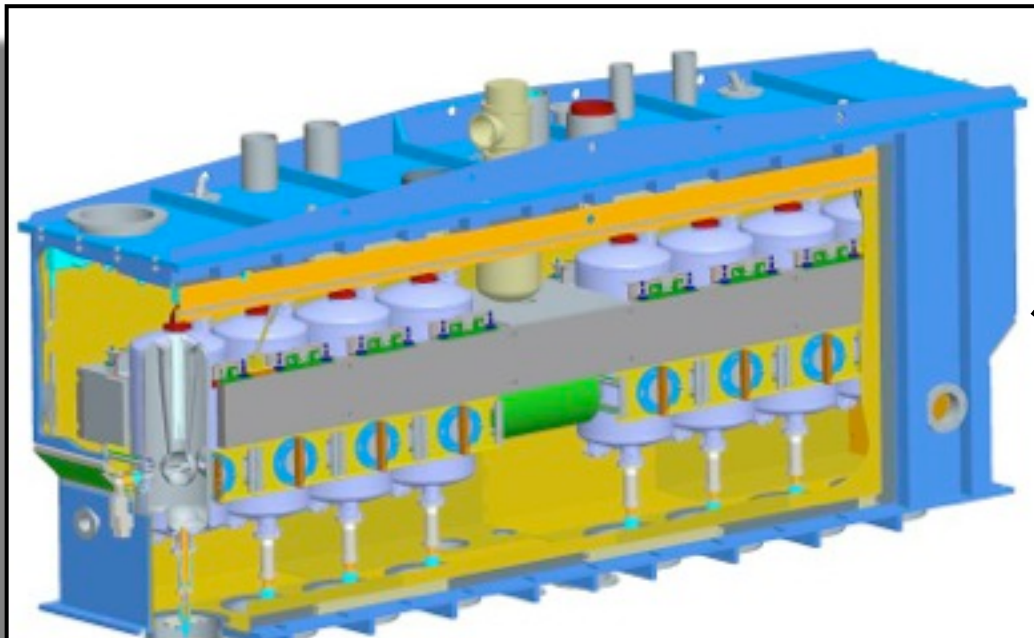
2° term: 2 gap effect → *d~βλ/2*

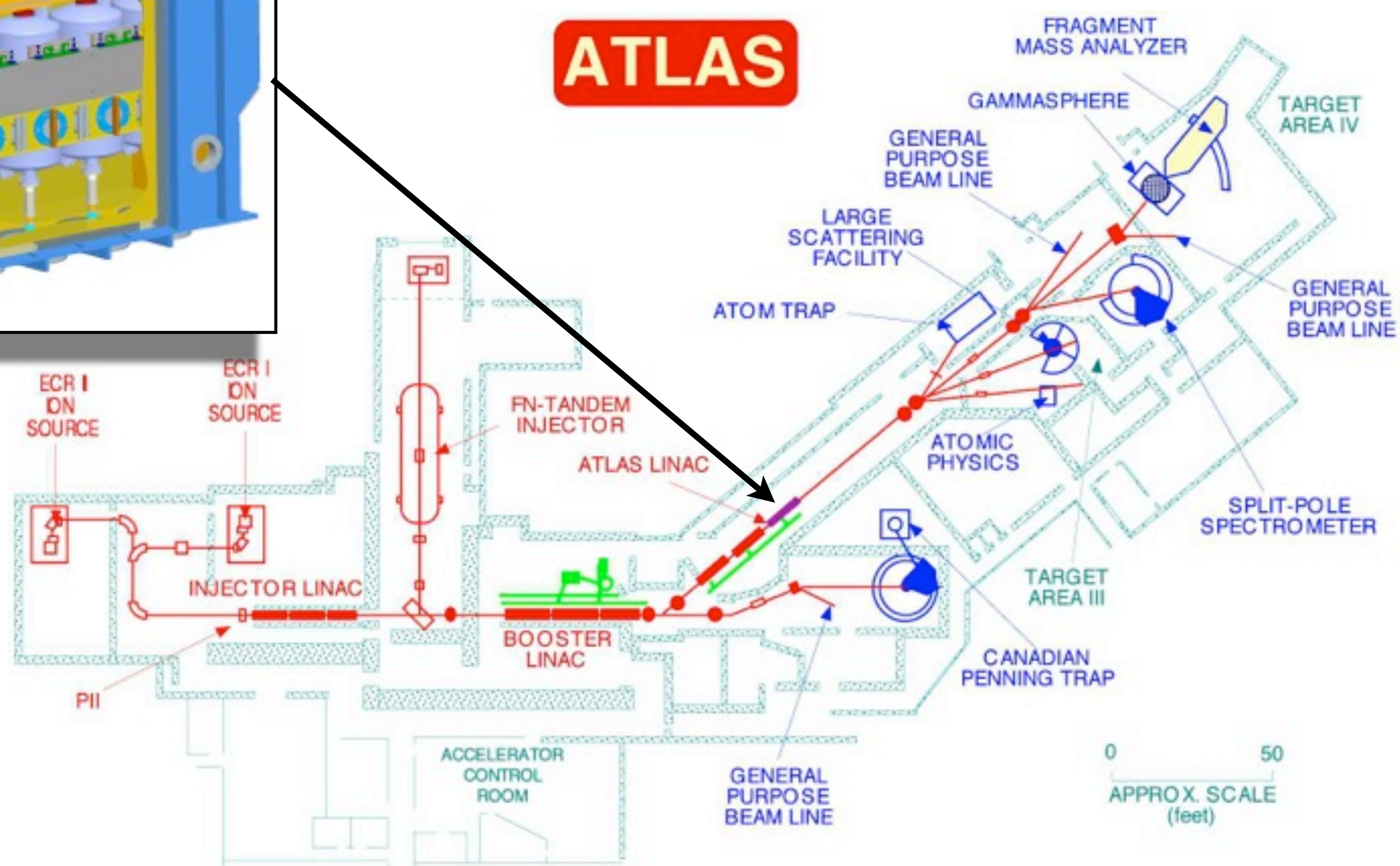
1°+ 2° term TTF curve

(For more than 2 equal gaps in π mode, the formulas change only in the 2° term)

Argonne Tandem Linac Accelerator System (ATLAS) at ANL

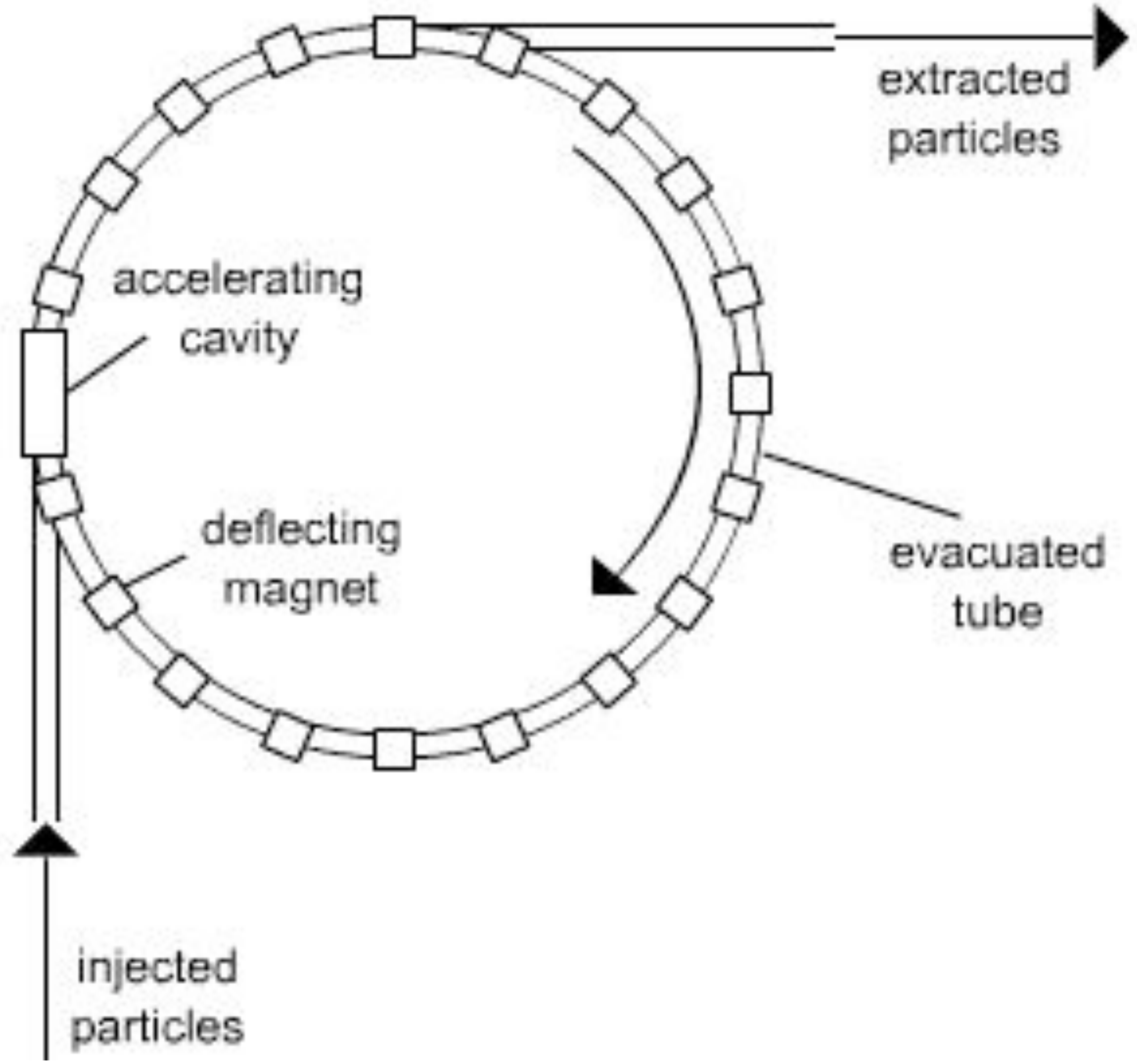
- Quarter-Wave Resonators inside a cryomodule





Back to Accelerator Story... The Synchrotron

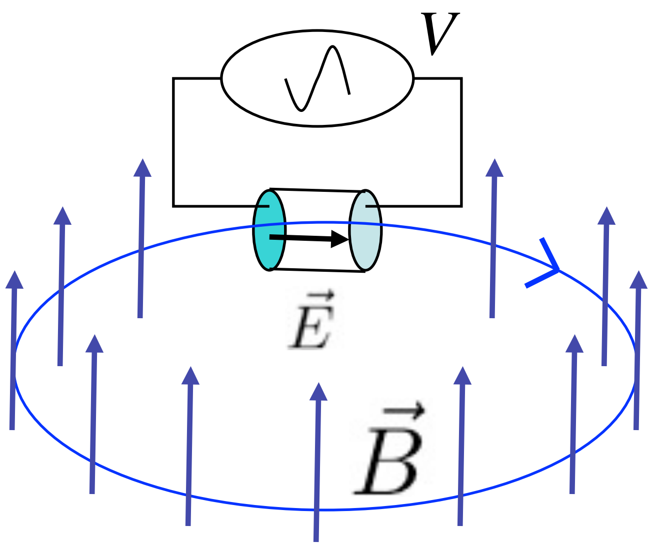
- Can achieve high energy at modest cost – tend to be used to deliver the highest energies
- Beam is accelerated in bunches
- Beam is accelerated internally and then ejected
- Intensity is limited by the Coulomb force of particles within a bunch (Space Charge)
- The magnets must ramp and this can be difficult to do quickly for superconducting magnets



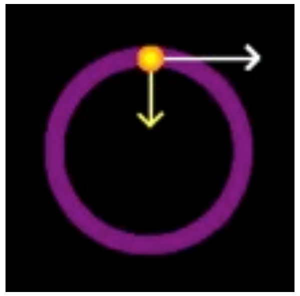
http://universe-review.ca/R15-20-accelerators.htm

Synchrotron (cont'd)

- Cavities used for acceleration; electromagnets used to guide in circular orbit. As magnetic field is increased, the time of arrival of the particle at the cavity will produce acceleration keeping the particle momentum in step with the magnetic field, thus keeping the orbit radius constant:



$$mv^2/R = evB \quad \Longrightarrow R = mv / eB$$
$$= p / eB$$



The quantity "$B \cdot \rho$" is called the *magnetic rigidity.*

$$B\rho = p/q \quad \approx \frac{10}{3} \text{ T-m} \cdot p_{[\mathrm{GeV}/c]}$$

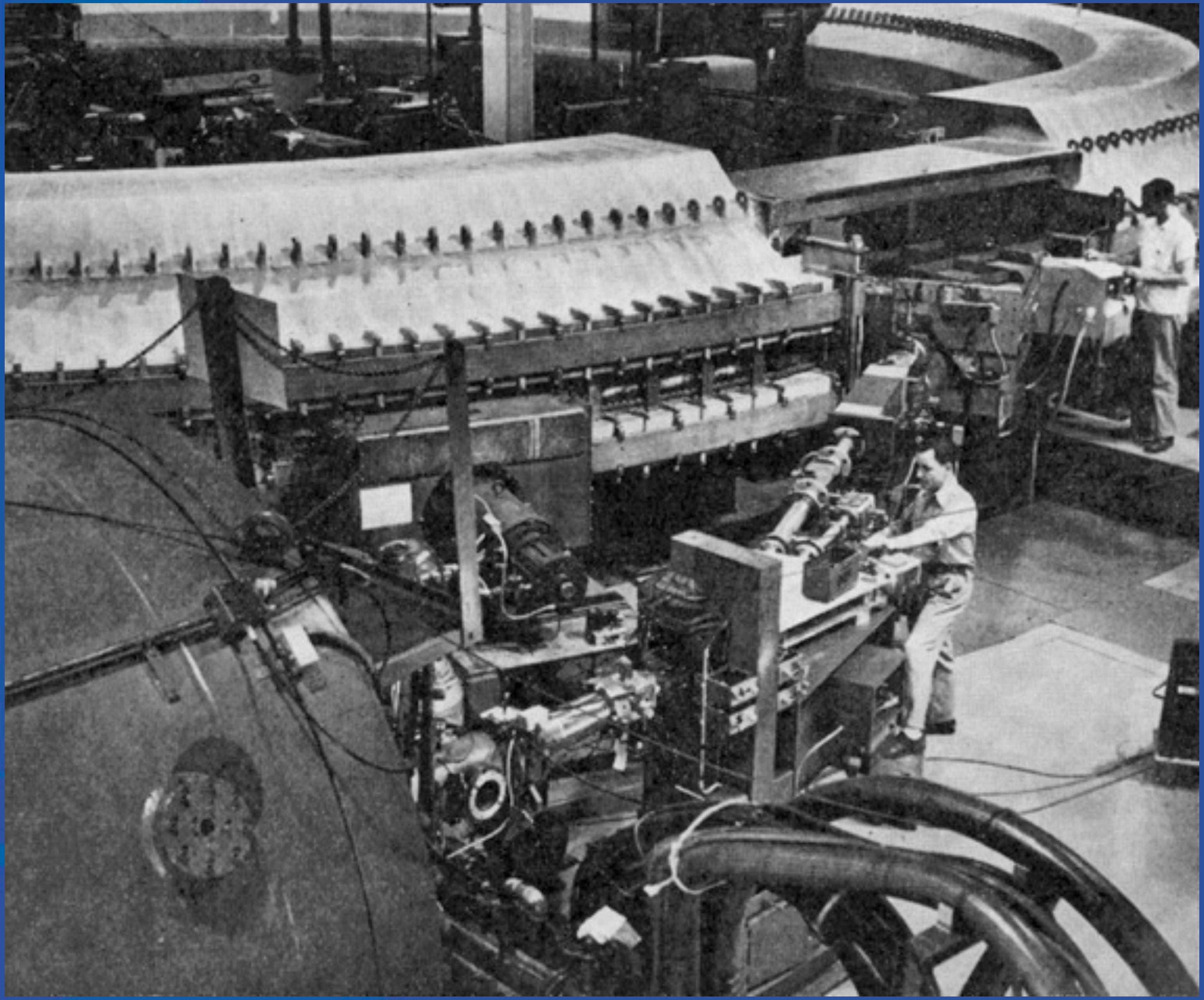
for a particle of charge e; divide by Q if charge is Qe.

What frequencies do we need?
Let's say v ~ c,
and say R = 1 m
then,
$f = v / 2\pi R$
$= (3 \times 10^8$ m/s$) / (2\pi$ 1m$)$
$= 5 \times 10^7$ / s = 50 MHz

FM Radio Stations: 88 - 108 MHz!
thus, we use RF cavities and power sources

The Synchrotron

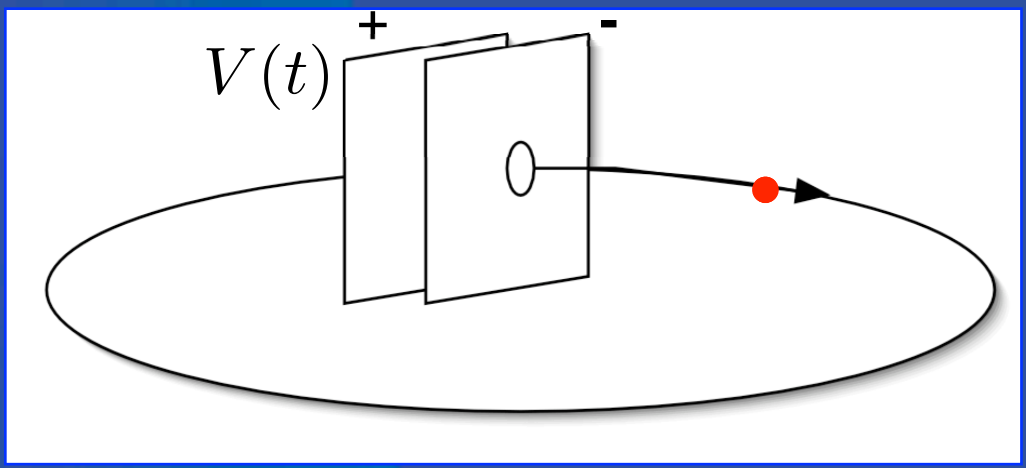
- 1st in U.S. was at G.E. research lab, late 1940's -- 70 MeV

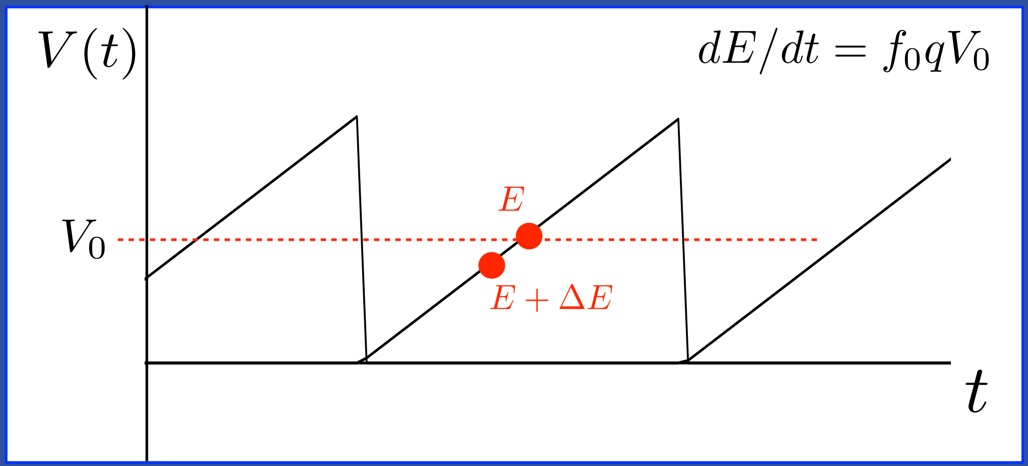


Cosmotron, 3.3 GeV
BNL (1952)

Accelerating Field

- Since the particles will have a distribution about the ideal energy of the accelerator, a "restoring force" is desired to keep particles nearby in energy as the "ideal" particle accelerates



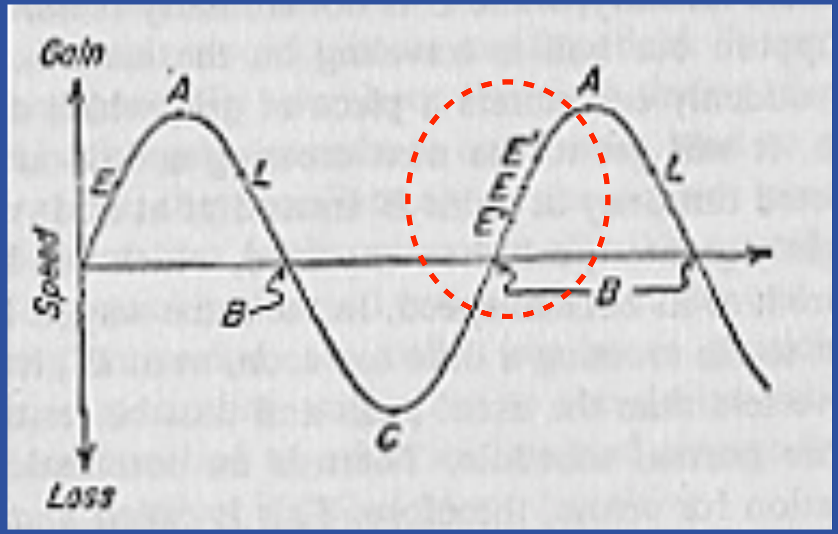


look at particles "near" the ideal

- If arrive late, gain energy; arrive early, get less
  - $\therefore$ restoring force $\Rightarrow$ stable energy oscillation possible

Introduction of a Nonlinearity

- Using a sinusoidal voltage for acceleration introduces a "non-linear" restoring force on the longitudinal oscillations



Here, ideal particle has energy E

  - *First studied by McMillan (U. Cal), and Veksler (Russia)*
- The non-linear field introduces...
  - oscillation frequency that varies with oscillation amplitude
  - stable and unstable fixed points
  - regions of stability (separatrices)

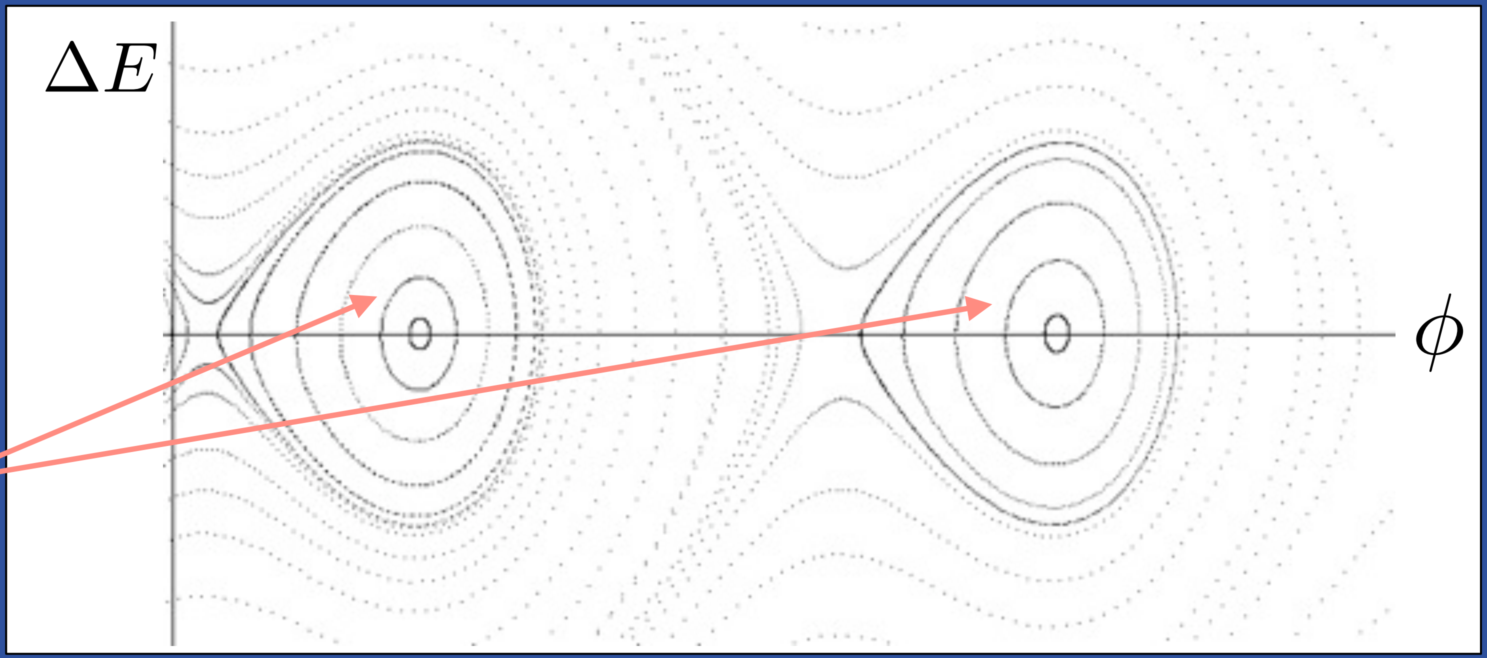
Fermilab

Synchrotrons...

If *ideal* particle has energy E_s and arrives at phase φ_s ...

particles arriving nearby in phase, and nearby in energy $E = E_s + \Delta E$ will oscillate about this ideal condition

Phase Space plot:

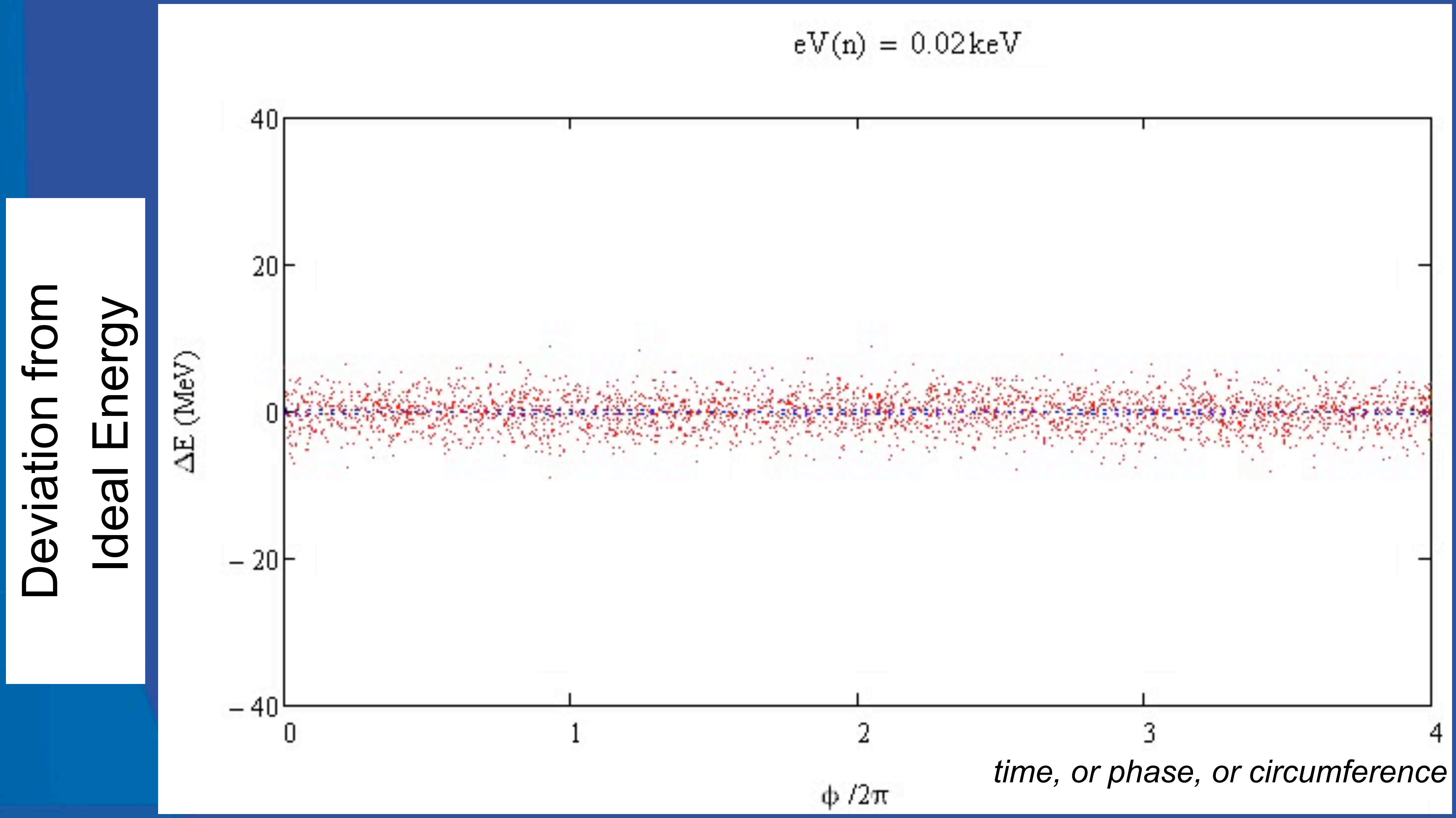


Stable Phase Space regions

- Adiabatic (on scale of energy oscillation period) increase of the magnetic field moves the stable fixed points; particles continue to oscillate, follow along

Creating a Bunched Beam

- Fill accelerator with particles, uniformly about the circumference; will have a natural spread in energies (~ < 1%, say)
- "Bunches" of particles are created by adiabatically raising voltage of RF cavities while magnetic field is held constant



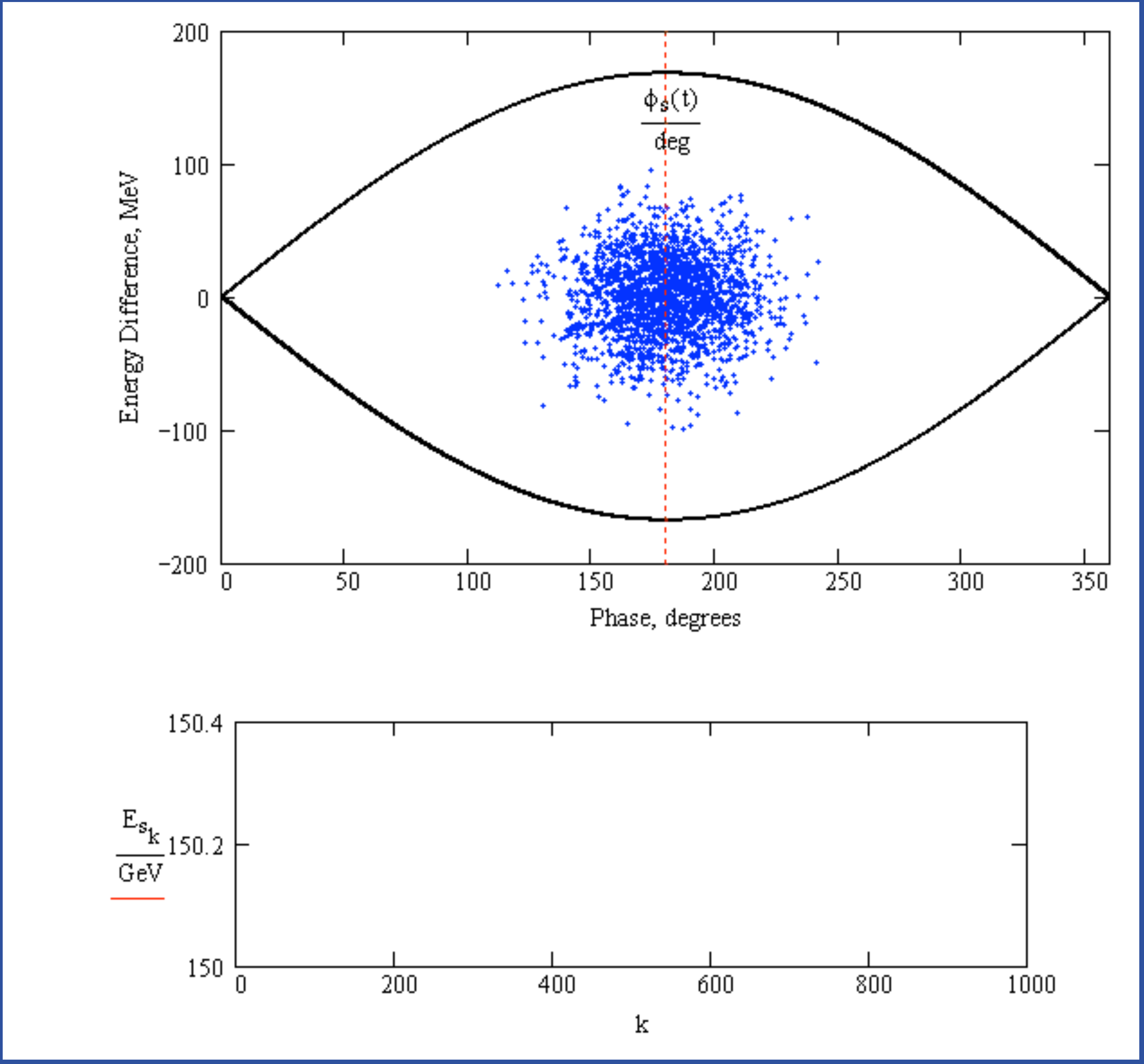
Acceleration -- Phase Stability

- Next, adiabatically raise the magnetic field
  - this alters the "synchronous phase"
    - $dE/dt = f_0\, qV \sin\varphi_s$

- Phase space area is an adiabatic invariant, and particles will "follow along"
- Oscillations about the "synchronous" momentum, defined by

  $p/q = B \cdot R$ for constant R :

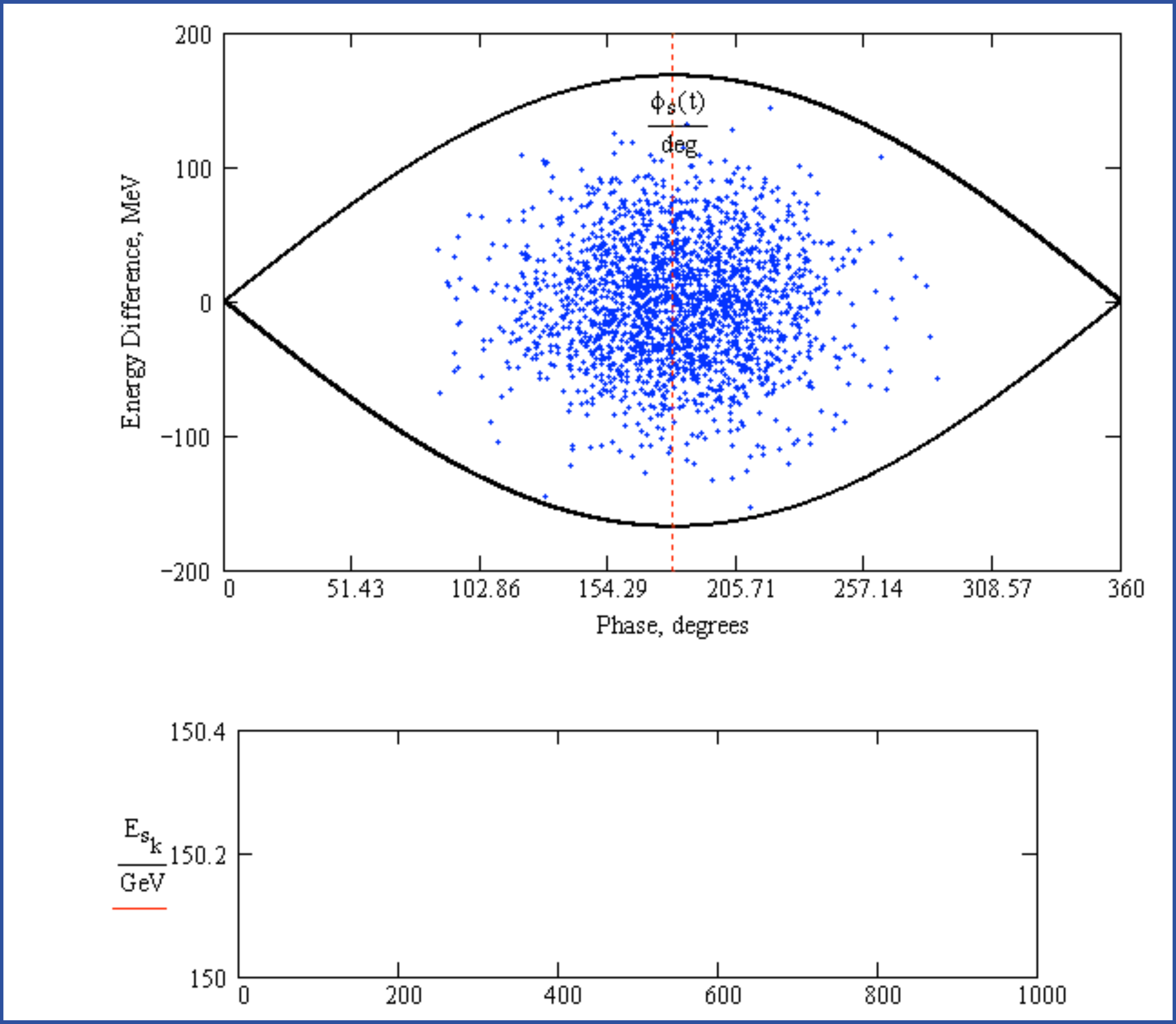
Synchrotron Oscillations



Fermilab

Acceleration -- Phase Stability

- If phase space area of the beam is too large, particle loss can occur during acceleration



Synchrotron Oscillations

Longitudinal Dynamics

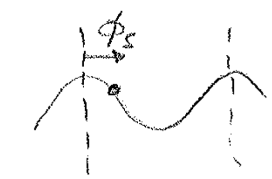
- The longitudinal (*Δz-Δp$_z$*, or *Δt-ΔE*) dynamics through the accelerator is essentially the same in synchrotrons and linacs
- Time between cavities: $\tau = L/v$
  - particles arrive at a phase with respect to the RF period which depends upon time to travel between cavities
  - this time can depend upon their speed *as well as* the path the particles take
    - » path length may depend upon speed as well -- through bending regions, for instance
- Equations of motion

ΔE_n, ϕ_n $\qquad$ ΔE_{n+1}, ϕ_{n+1}

$$\boxed{\begin{aligned} \phi_{n+1} &= \phi_n + \frac{2\pi h \eta}{\beta^2 E}\,\Delta E_n \\ \Delta E_{n+1} &= \Delta E_n + \frac{Q}{A}\, eV\,(\cos\phi_{n+1} - \cos\phi_s) \end{aligned}}$$

notes:

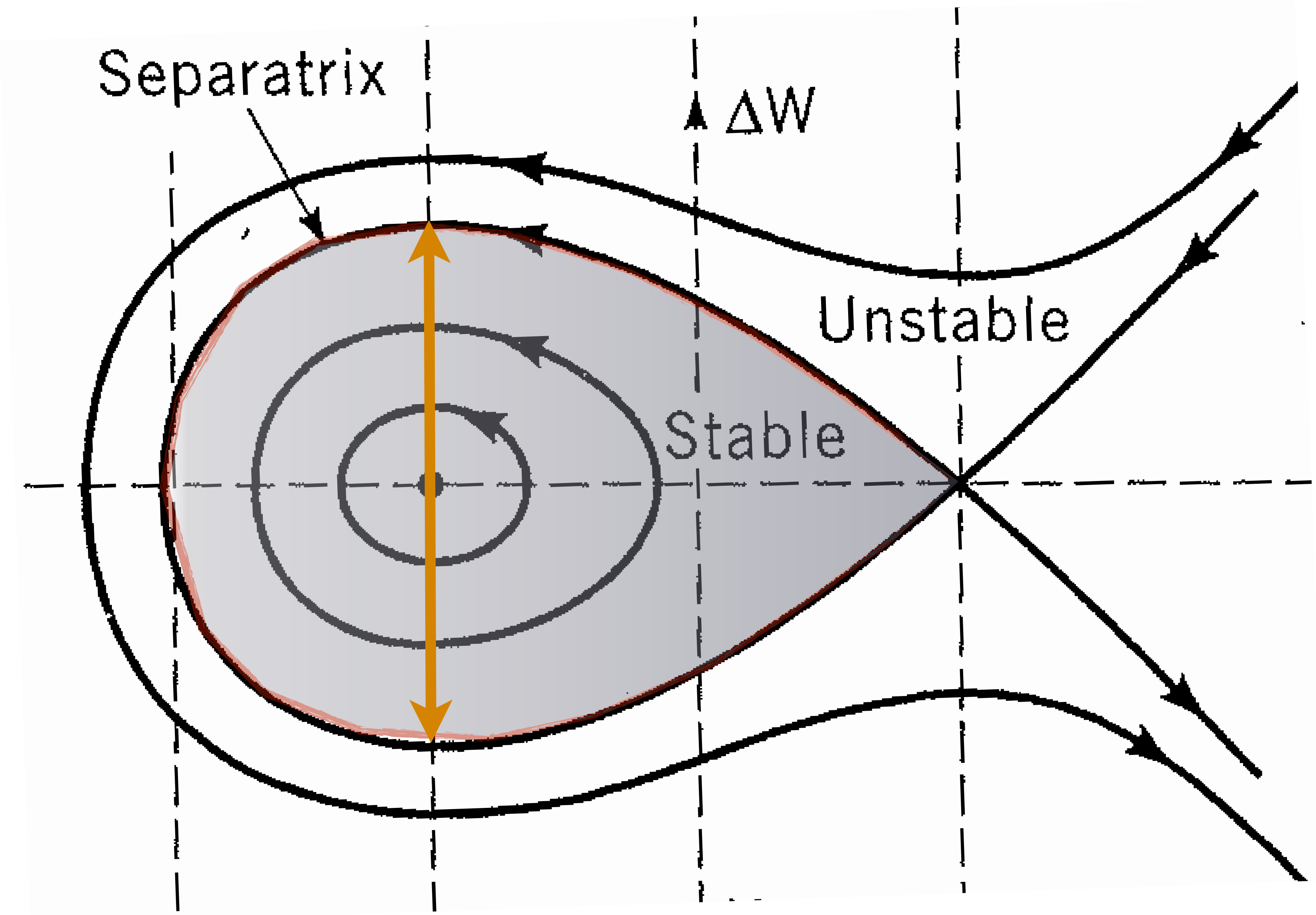
- $eV \Rightarrow V$ = voltage amplitude, including TTF(β) $\qquad h = L/\beta\lambda$, $\lambda = c/f_{RF}$ $\qquad$ (difference equations)
- $\eta = (\Delta\tau/\tau)/(\Delta p/p) = \dfrac{\Delta L/L}{\Delta p/p} - \dfrac{1}{\gamma^2}$



- as written here, E, ΔE are "per nucleon"
- $E = mc^2 + W$, $\quad mc^2 = 931$ MeV, $\quad \Delta E \leftrightarrow \Delta W$

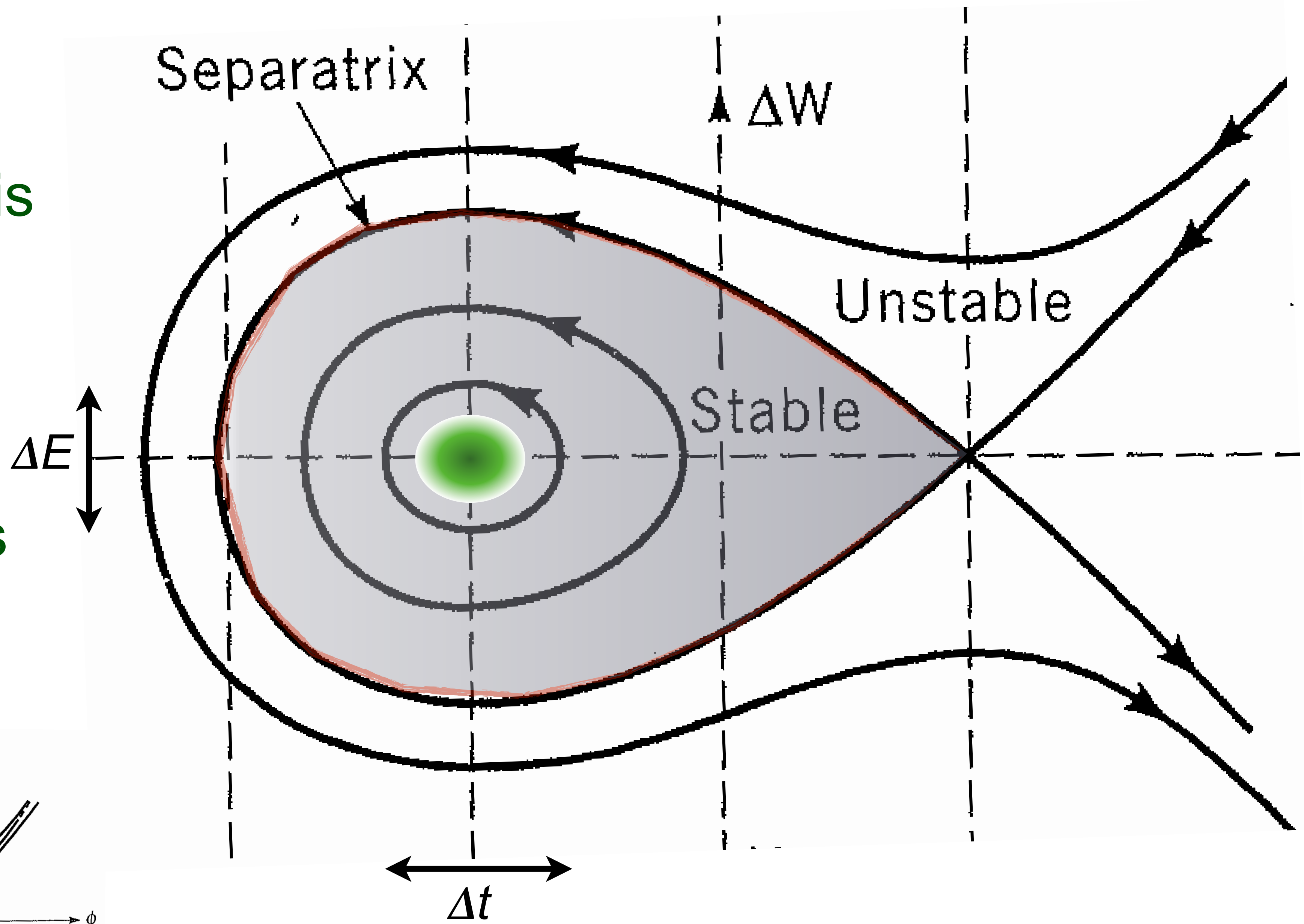
Acceptance and Emittance

- Stable region often called an RF "bucket"
  - "contains" the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system

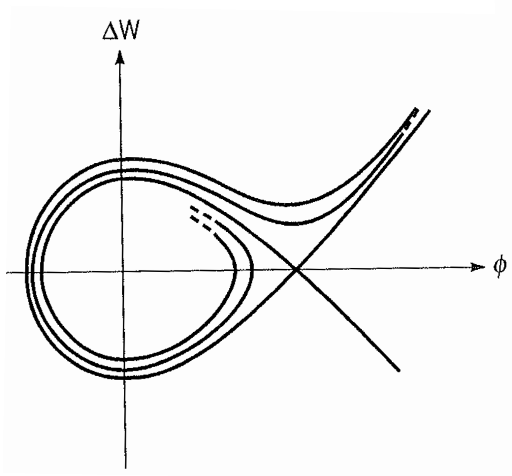


Acceptance and Emittance

- Stable region often called an RF "bucket"
  - "contains" the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
- Desire the beam particles to occupy much smaller area in the phase space



In linacs, fractional energy change can be large, and so this will distort the phase space



area: "eV-sec"
Note: *E, t canonical*

Stability of Longitudinal Motion

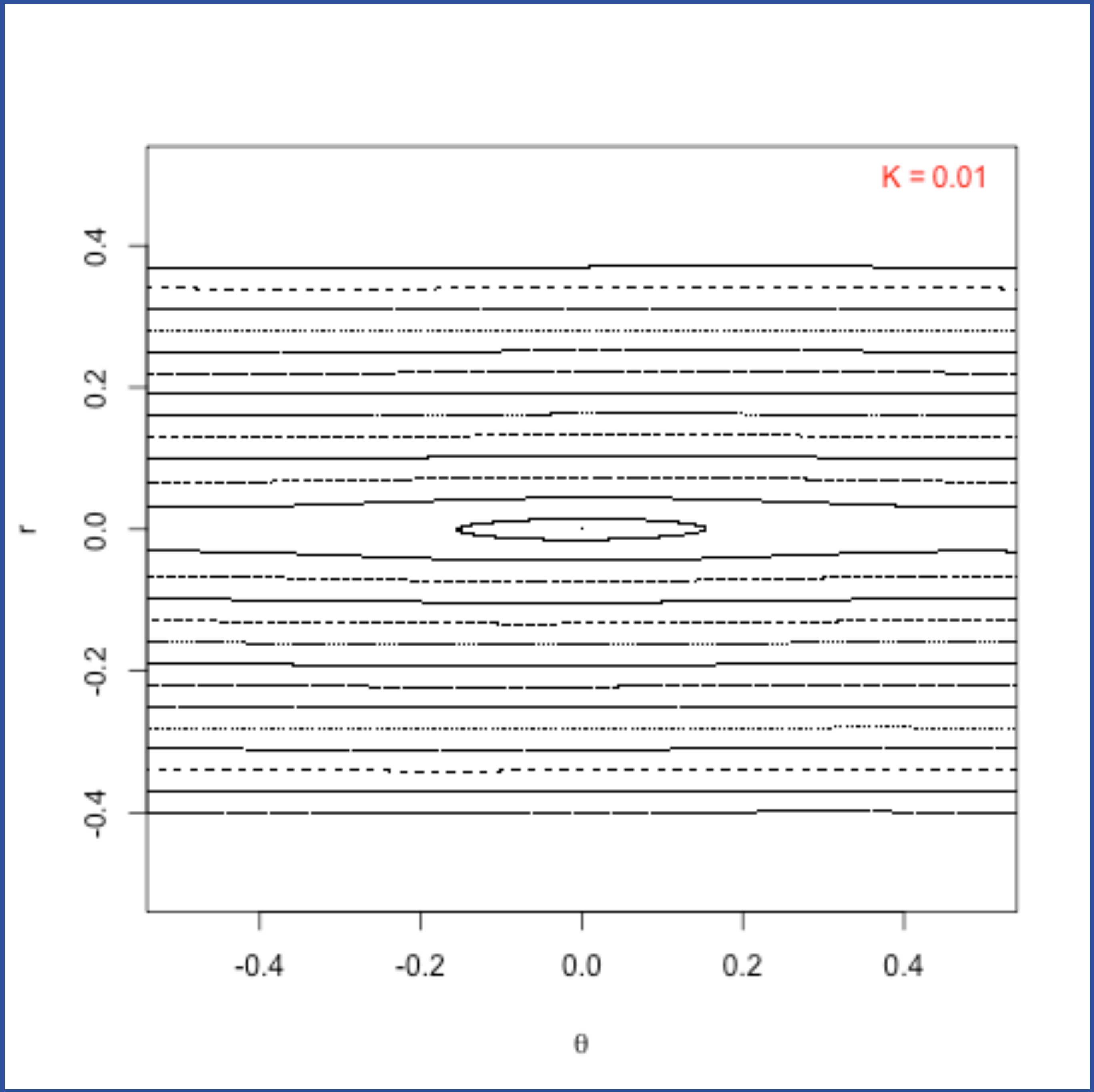
- If longitudinal motion is "slow", can typically treat time as differential variable
- However, acceleration happens at discrete "points" in the linac or synchrotron; perhaps more accurate to treat as a "map":

$$
\begin{aligned}
\Delta E_{n+1} &= \Delta E_n + eV(\sin\omega_{\mathrm{rf}}\Delta t_n - \sin\phi_s) \\
\Delta t_{n+1} &= \Delta t_n + k\,\Delta E_{n+1}
\end{aligned}
$$

- Essentially the "Standard Map" (when $\phi_s = 0$)
  - (or Chirikov-Taylor map, or Chirikov standard map)

$$
\begin{aligned}
p_{n+1} &= p_n - K\sin\theta_n \\
\theta_{n+1} &= \theta_n + p_{n+1}
\end{aligned}
$$

Phase Space of the Standard Map



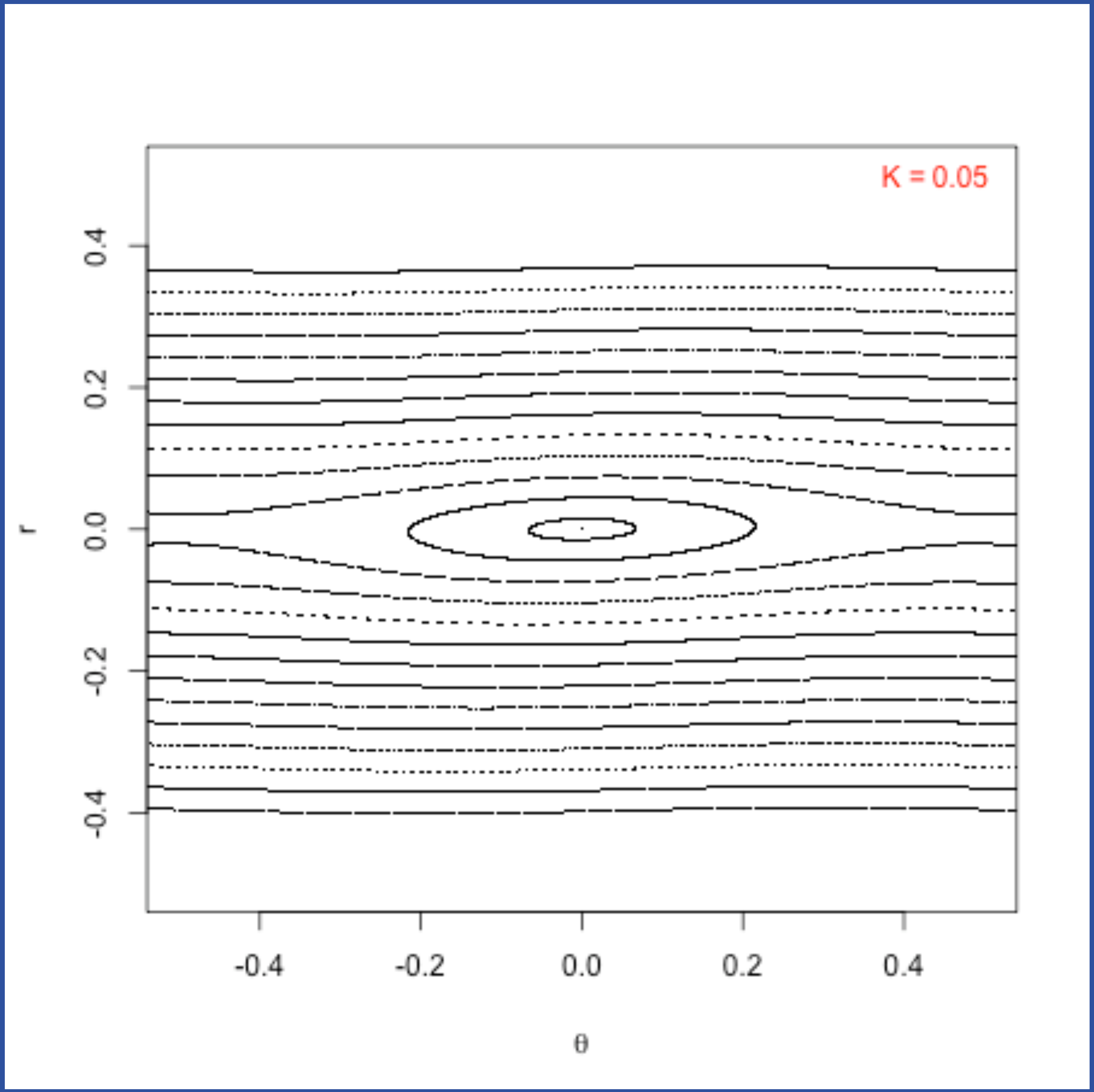
Each view uses the same initial conditions for 27 particles

Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Phase Space of the Standard Map



Each view uses the same initial conditions for 27 particles

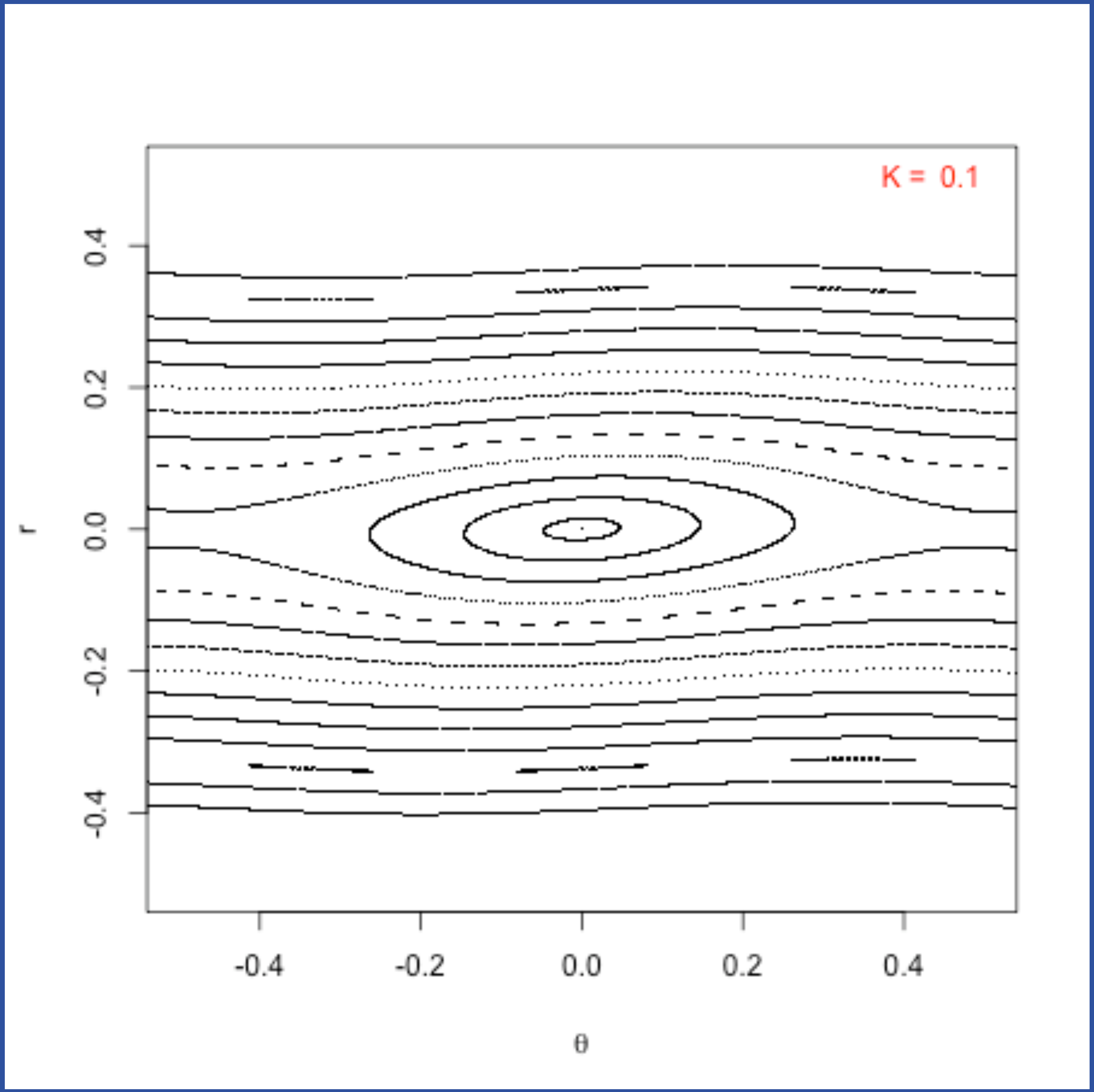
Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Fermilab

Phase Space of the Standard Map



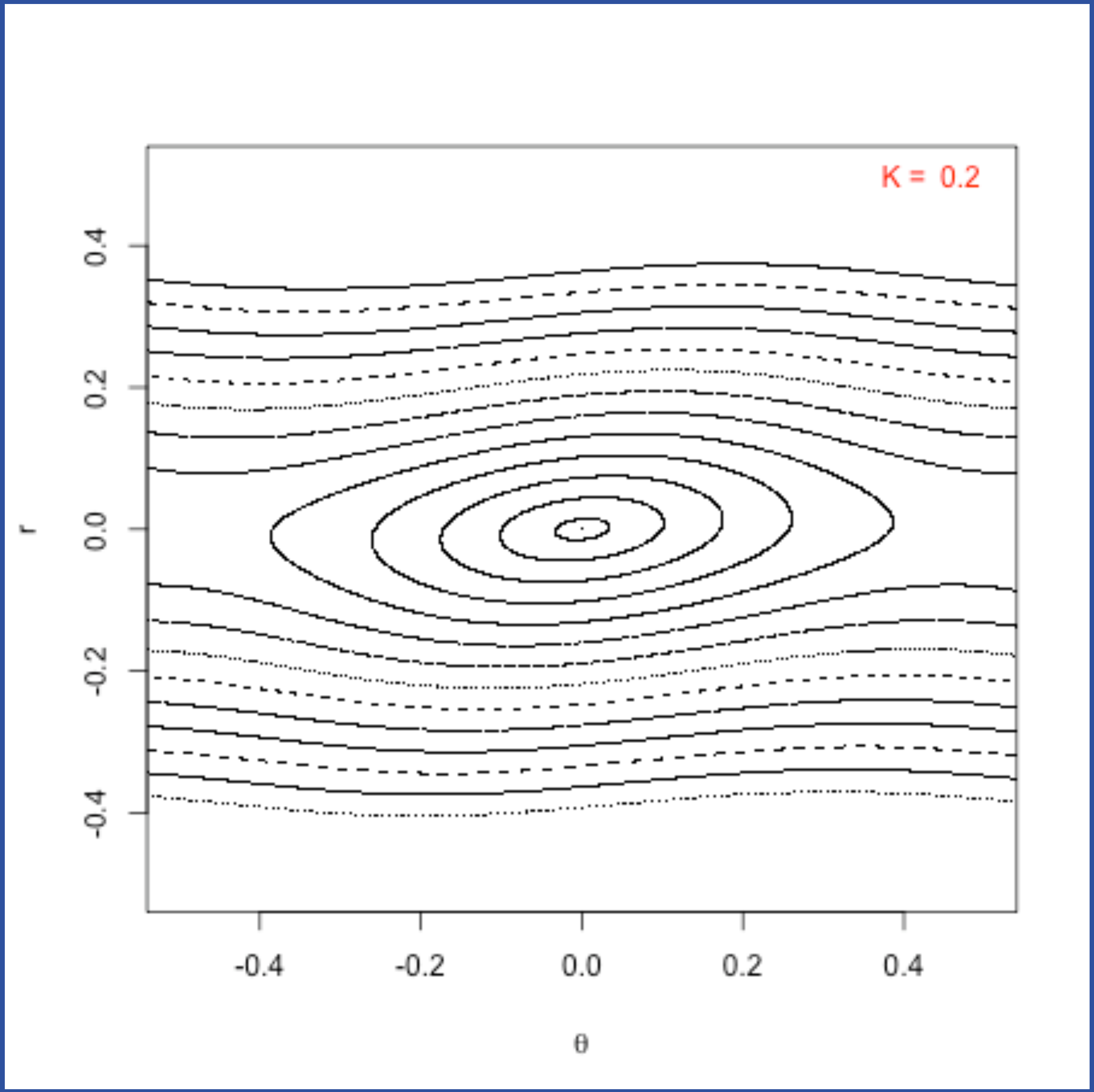
Each view uses the same initial conditions for 27 particles

Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Phase Space of the Standard Map



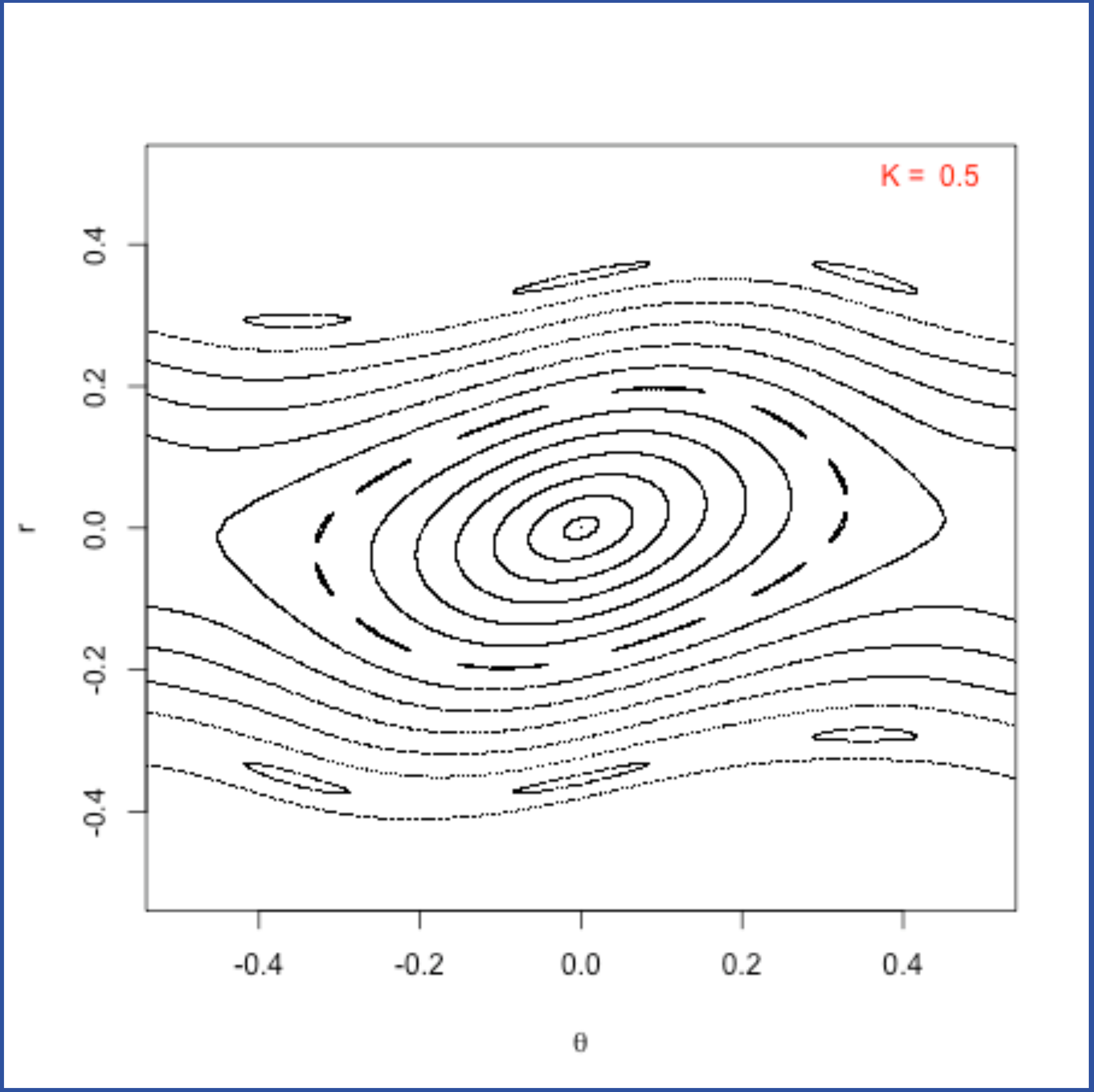
Each view uses the same initial conditions for 27 particles

Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Phase Space of the Standard Map



Each view uses the same initial conditions for 27 particles

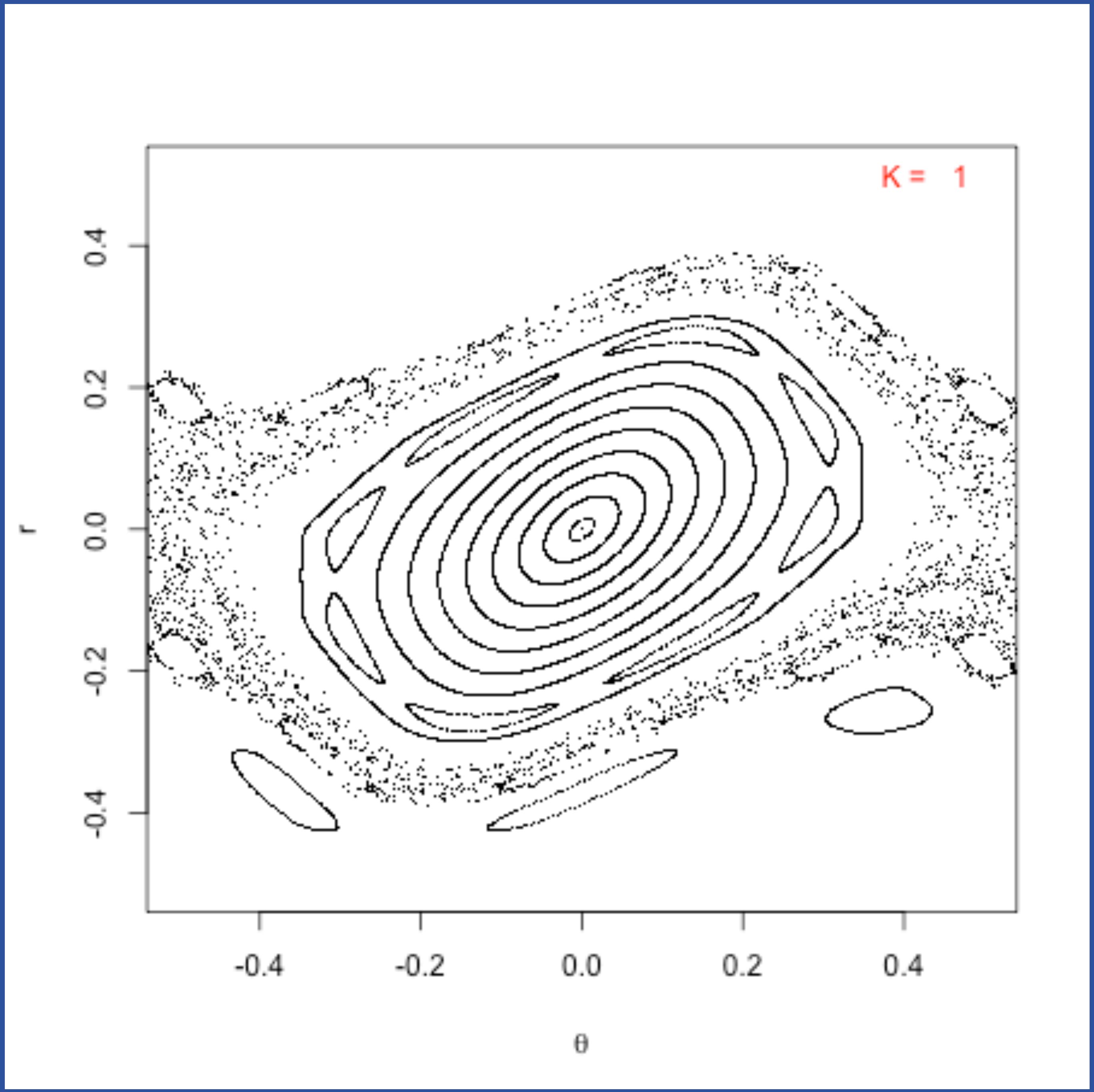
Typical linacs and synchrotrons:

K ~ 0.0001 - 0.1

- A Limit of Stability?... can analyze later

Fermilab

Phase Space of the Standard Map



Each view uses the same initial conditions for 27 particles

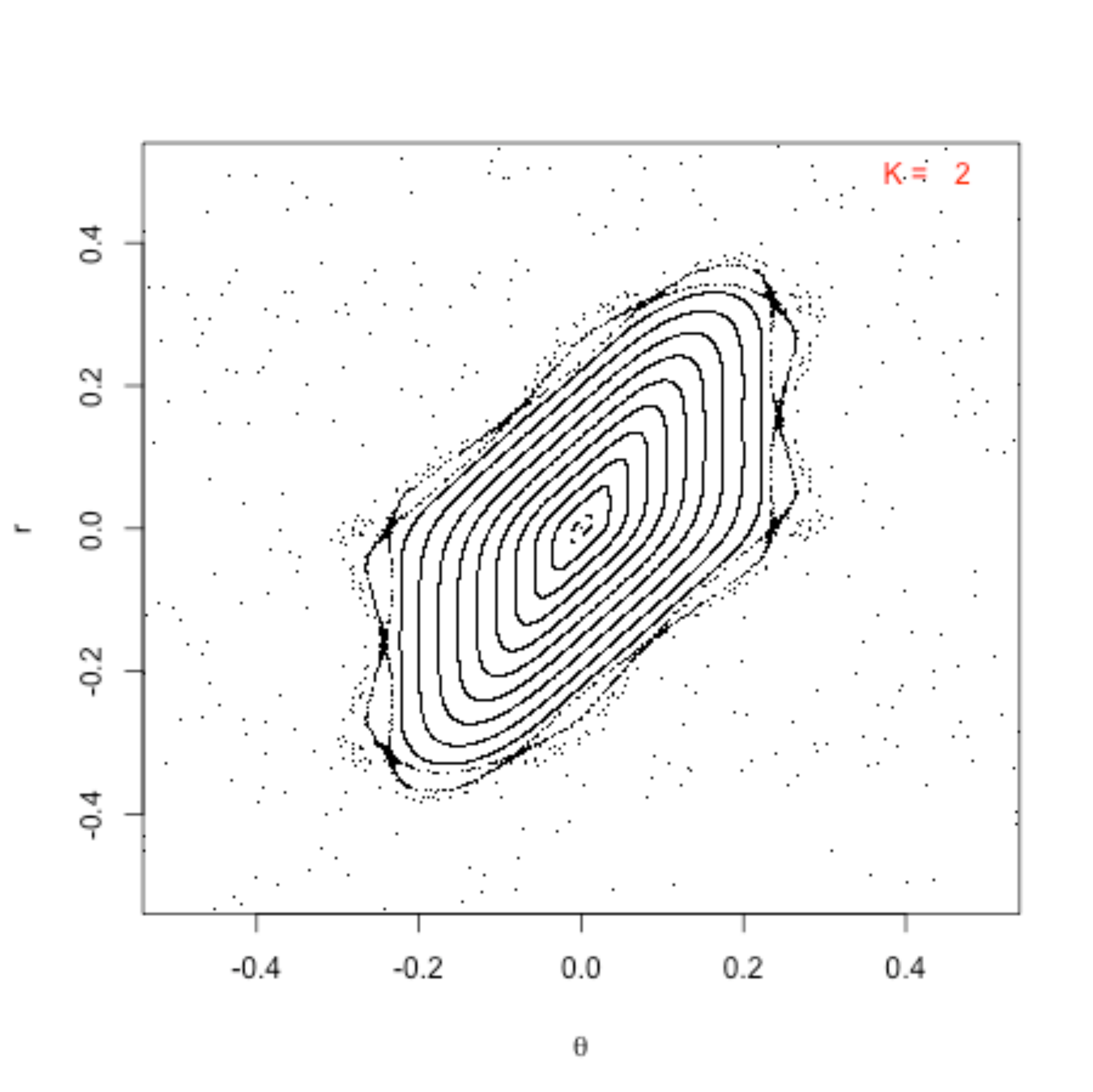
Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Fermilab

Phase Space of the Standard Map



Each view uses the same initial conditions for 27 particles

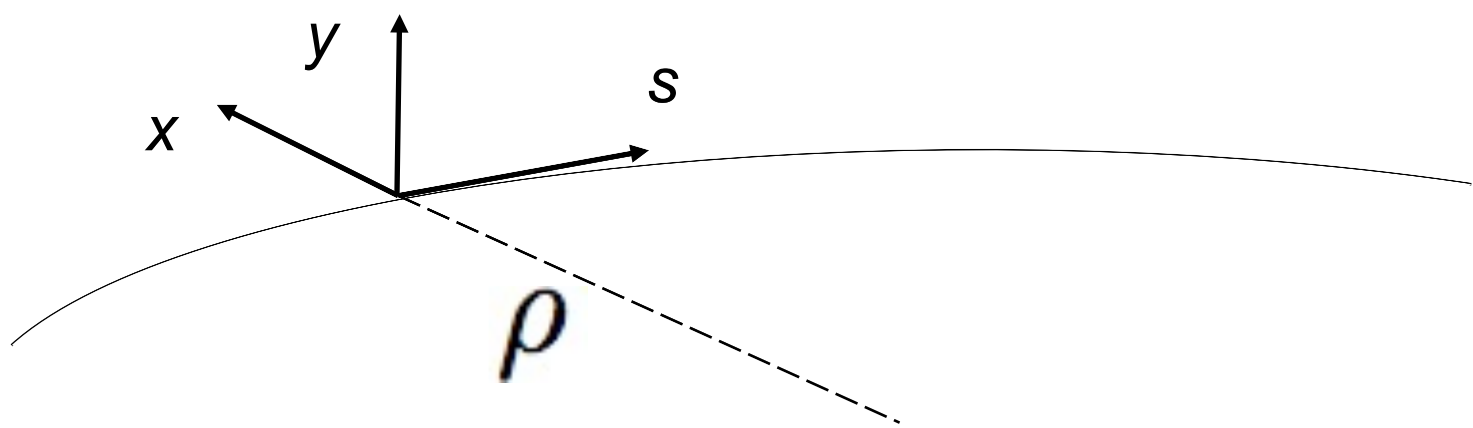
Typical linacs and synchrotrons:

$K \sim 0.0001 - 0.1$

- A Limit of Stability?... can analyze later

Transverse Equations of Motion

- Need for Transverse Focusing
  - not all particles (any?) begin "on" the design trajectory
  - need to keep particles nearby
- Reference Trajectory and Local Coordinate System
  - Electrostatic -- low energy
  - Magnetic -- high energy



- Lorentz Force $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$

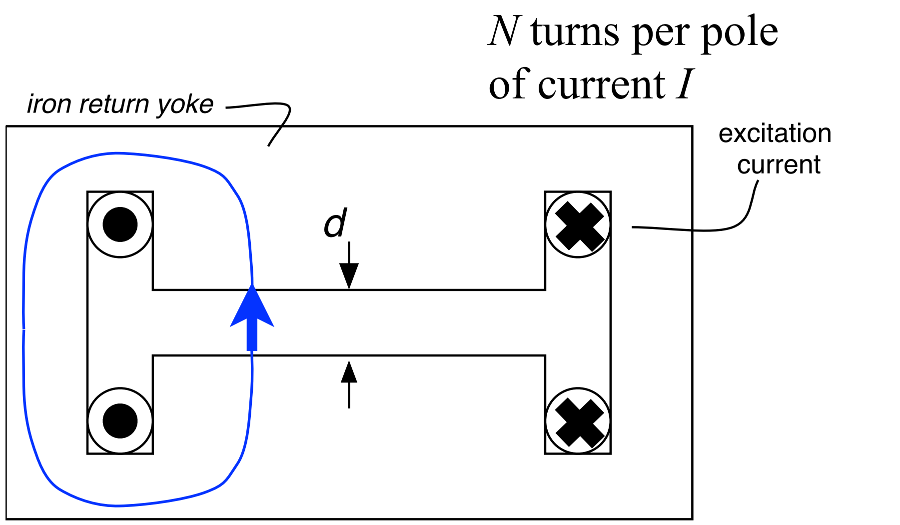
- Magnetic Rigidity
  - (ion w/ mass A, charge Q)

$$B\rho = \frac{A}{Q}\left(\frac{1}{300}\frac{\mathrm{T}\cdot\mathrm{m}}{\mathrm{MeV/c/u}}\right)p$$

Magnets

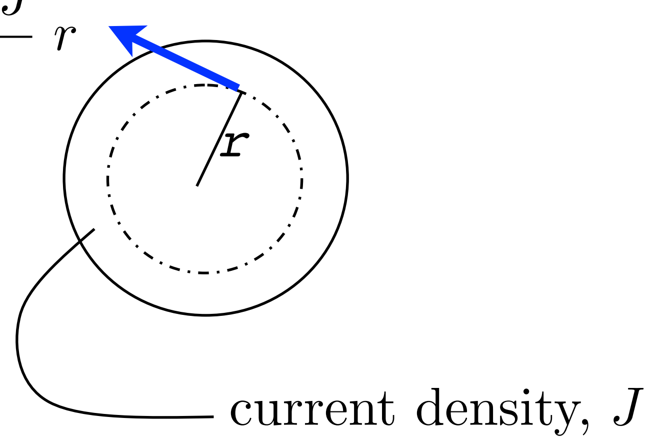
- Iron-dominated magnetic fields
  - iron will "saturate" at about 2 Tesla

$$B = \frac{2\mu_0 N \cdot I}{d}$$



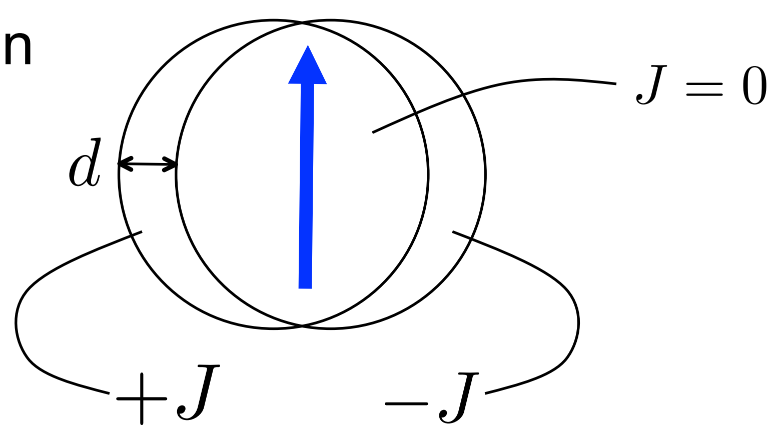
- Superconducting magnets
  - field determined by distribution of currents

$$B_\theta = \frac{\mu_0 J}{2} r$$



"Cosine-theta" distribution

$$\boxed{B_x = 0, \quad B_y = \frac{\mu_0 J}{2} d}$$



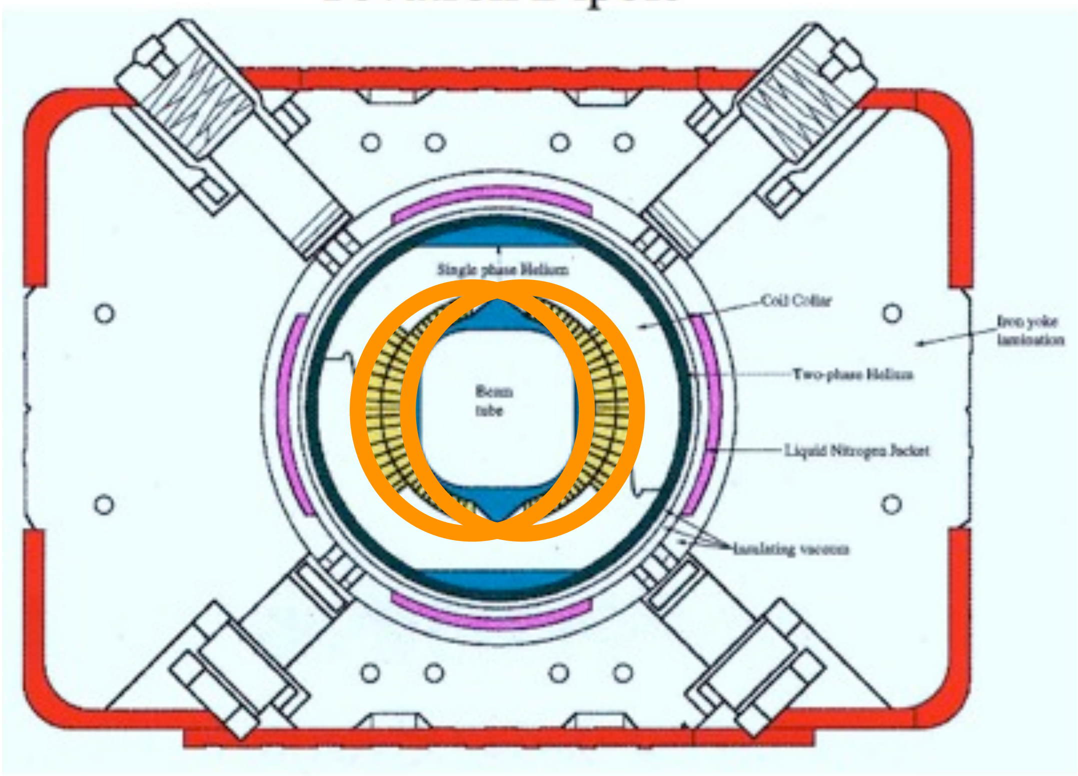
High-Field Superconducting Designs

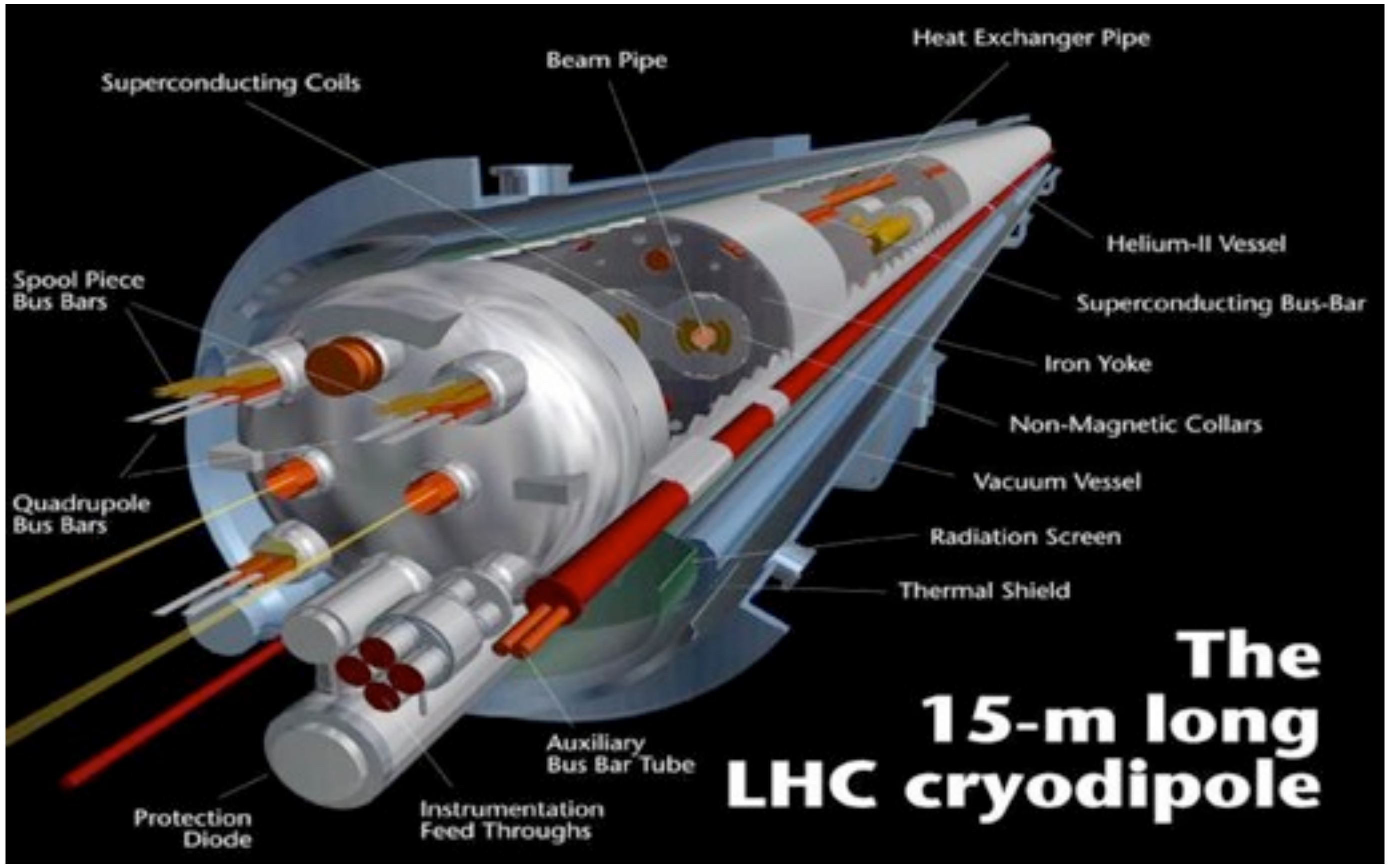
Numerical Example:

$$\begin{aligned} B &= \frac{\mu_0 J}{2} d \\ &= \frac{4\pi\ \mathrm{T\ m/A}}{10^7}\,\frac{1000\ \mathrm{A/mm^2}}{2}\cdot(10\ \mathrm{mm})\cdot\frac{10^3\mathrm{mm}}{\mathrm{m}} \\ &= 6\ \mathrm{T} \end{aligned}$$

- Tevatron
  - 1st SC accelerator
  - 4.4 T; 4°K

Tevatron Dipole

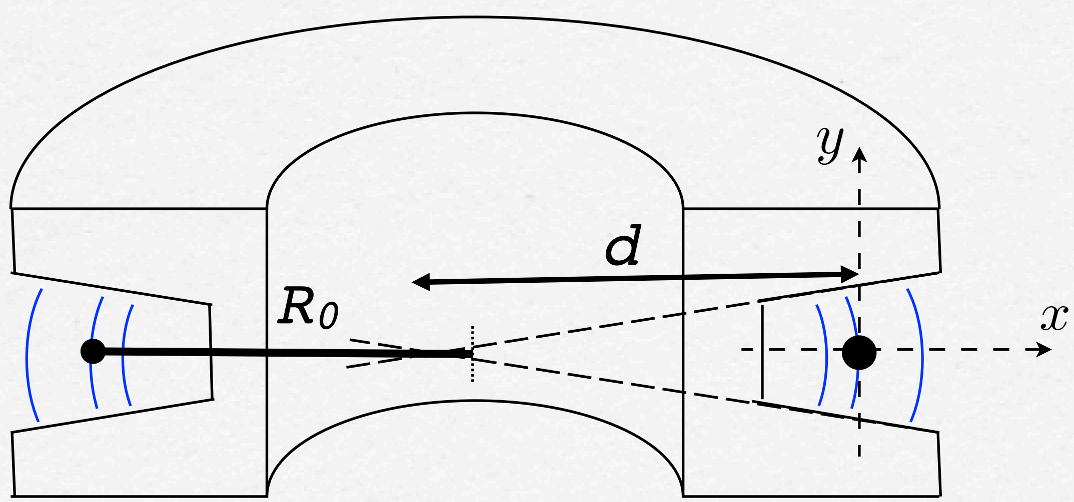




- LHC -- 8 T; 1.8°K

Keeping Focused

- In addition to increasing the particle's energy, must keep the beam focused transversely along its journey
- Early circular accelerators used what is now called "weak focusing"



$$B = B_0 \left(\frac{R_0}{r}\right)^n$$

n is determined by adjusting the opening angle between the poles

$$n \approx \frac{R_0}{d}$$

must have $0 \leq n \leq 1$ for stability

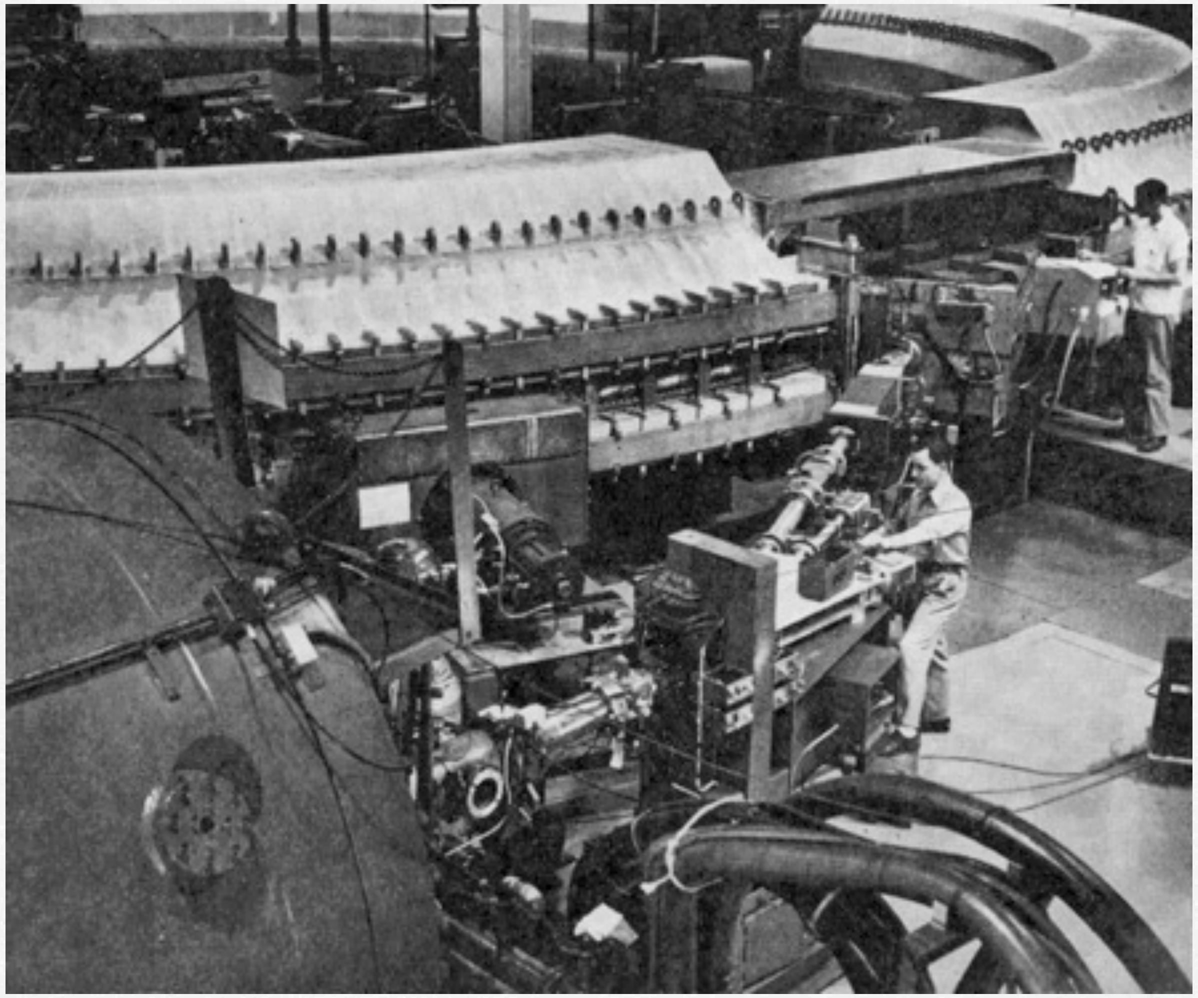
$d = \infty,\ n = 0$

$d = R_0,\ n = 1$

Room for improvement...

- With weak focusing, for a given transverse angular deflection,
- Thus, aperture ~ radius ~ energy

$$x_{max} \sim \frac{R_0}{\sqrt{n}}\,\theta$$



Cosmotron (1952)

(3.3 GeV)

Room for improvement...

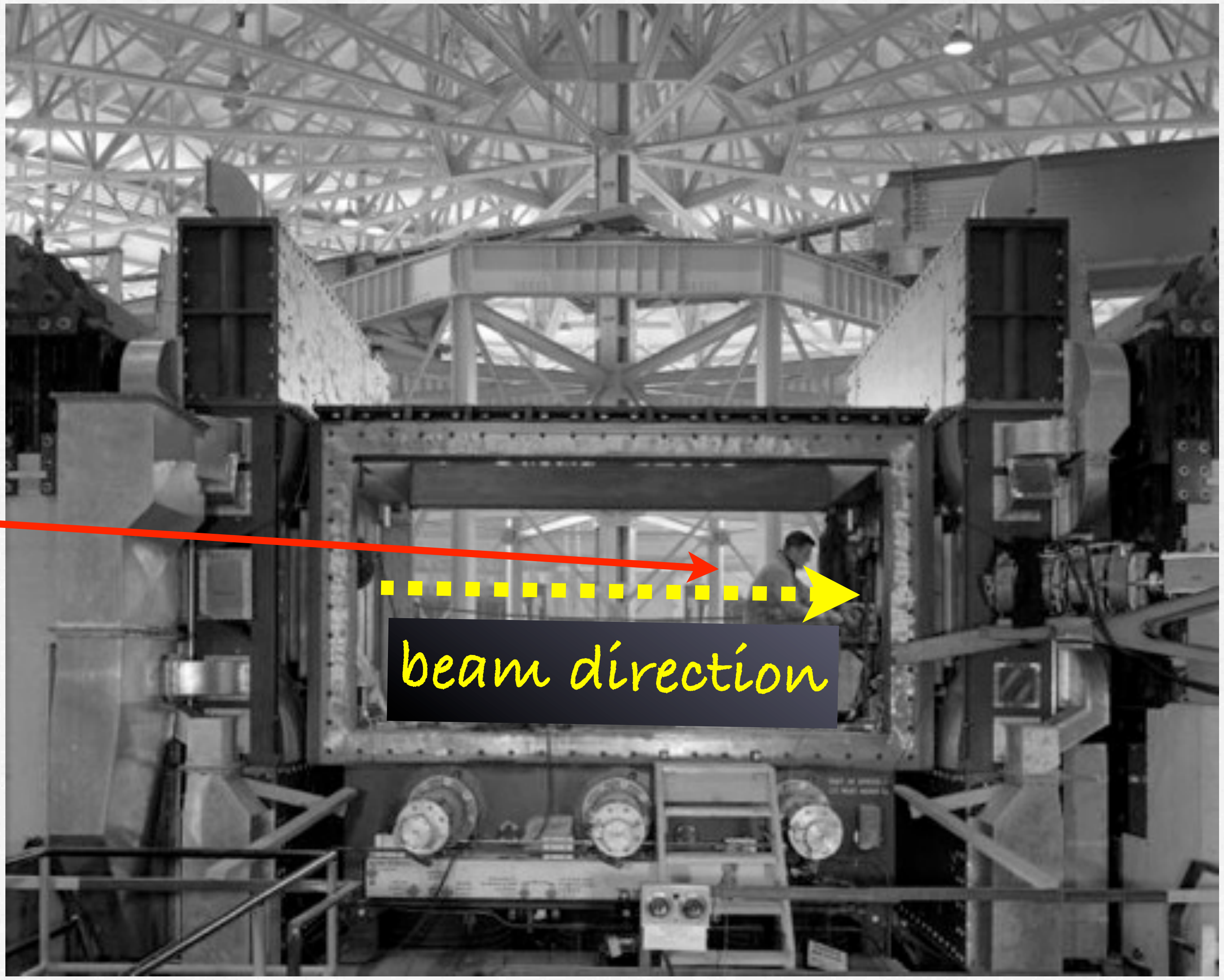
- With weak focusing, for a given transverse angular deflection,
- Thus, aperture ~ radius ~ energy

$$x_{max} \sim \frac{R_0}{\sqrt{n}}\,\theta$$

Bevatron (1954)

(6 GeV)

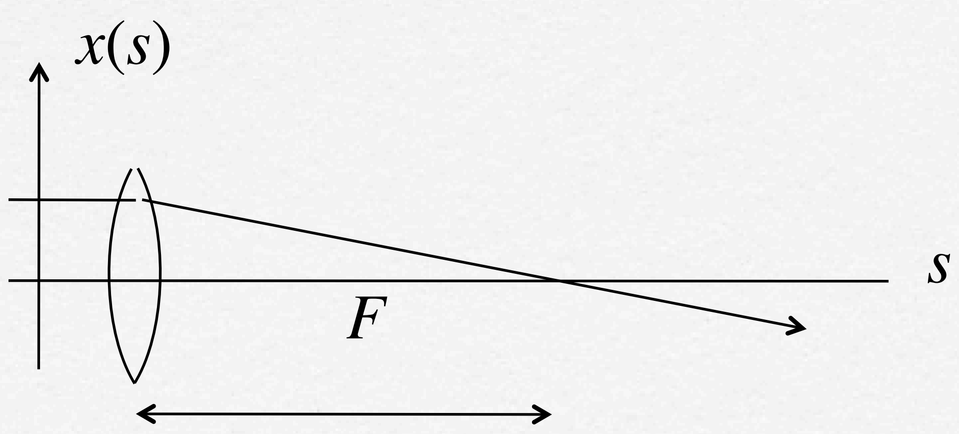
Could actually sit inside the vacuum chamber!!

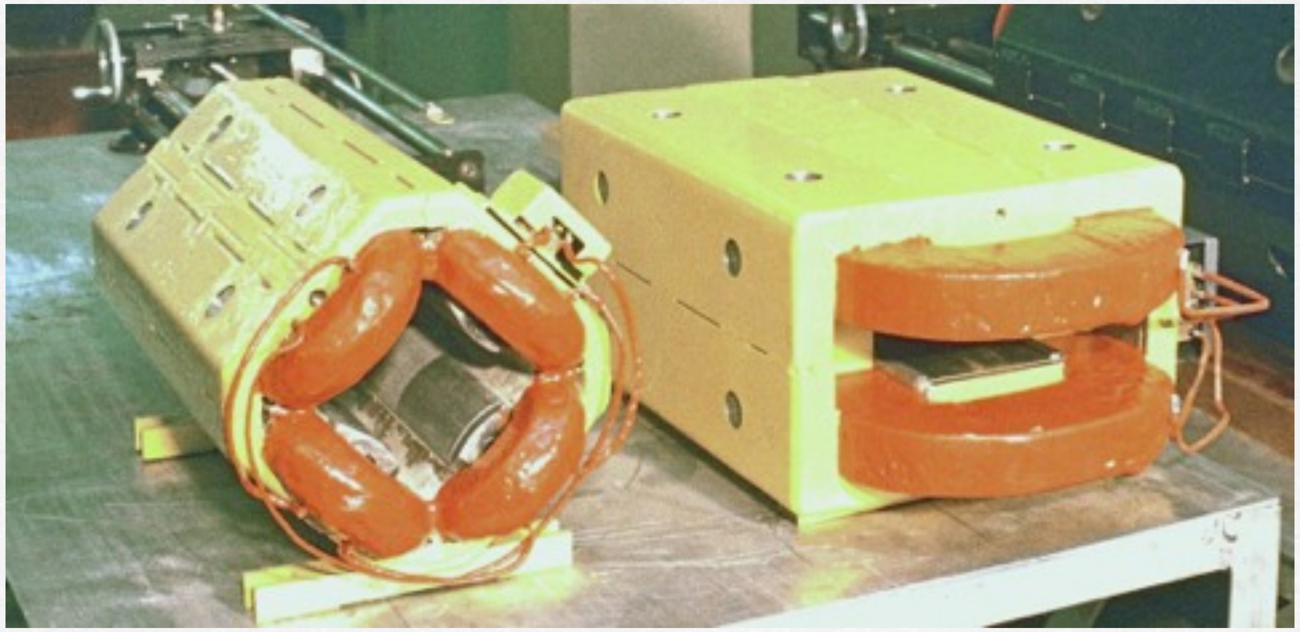


Separated Function

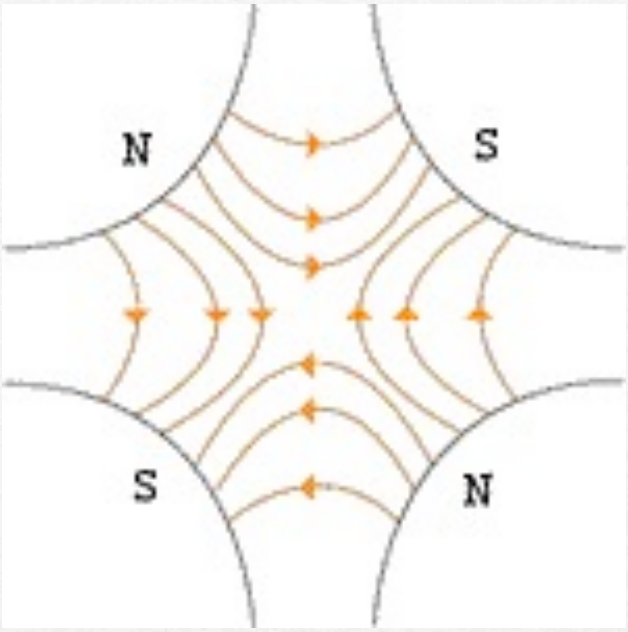
- Until late 60's, early accelerator magnets (wedge-shaped variety) both focused and steered the particles in a circle. ("combined function")
- Now, use "dipole" magnets to steer, and use "quadrupole" magnets to focus
- Quadrupole magnets, with alternating field gradients, "focus" particles about the central trajectory -- act like lenses
- Thin lens focal length:

$$\Delta x' = eB_y\ell/p = (eB'\ell/p)\,x \rightarrow 1/F = eB'\ell/p$$



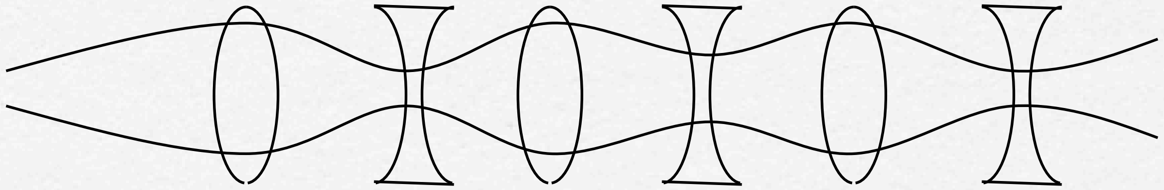


Strong Focusing



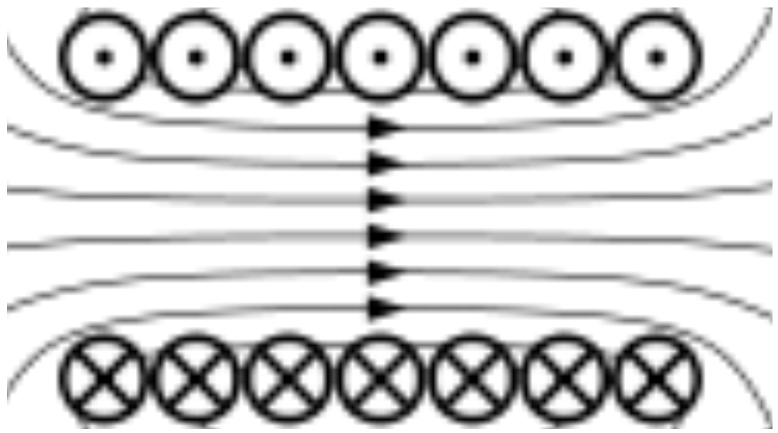
- Think of standard focusing scheme as alternating system of focusing and defocusing lenses (today, use quadrupole magnets)

- Quadrupole will focus in one transverse plane, but defocus in other; if alternate, can have net focusing in both
  - alternating gradients:



Solenoid Focusing

- Solenoid Field



$$B = \mu_0 N' I$$

FRIB linac solenoids ~ 9 T

- Particle trajectory in a uniform field

Helical, with radius $a = mv_\perp/(qB_0)$

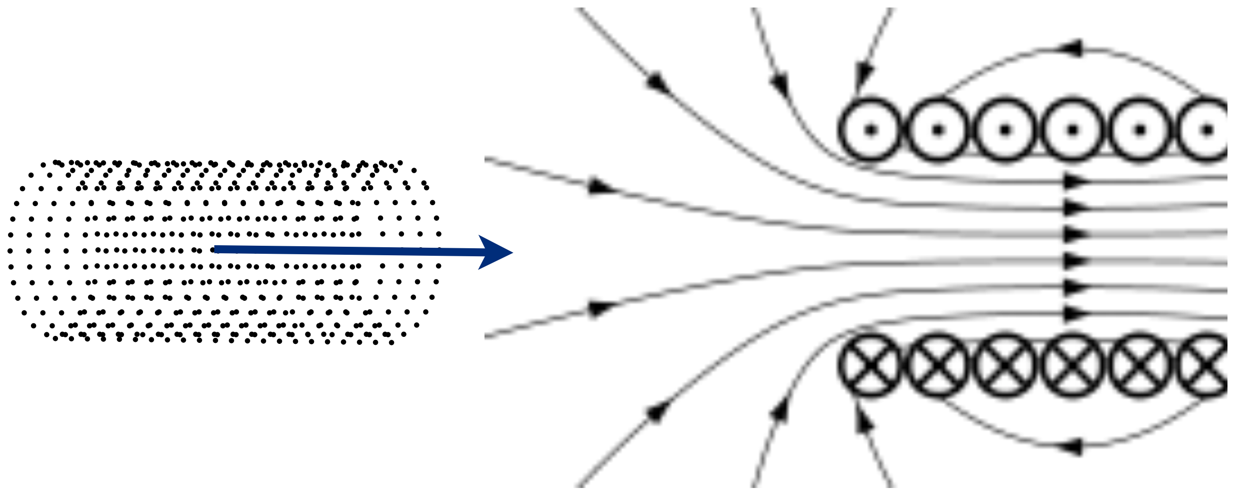
where $v_\perp$ is the velocity perpendicular to $\vec{B}$

$$\omega = \frac{v_\perp}{a} = qB_0/m$$

- So, how does a solenoid "focus"?

Acquisition of Angular Momentum

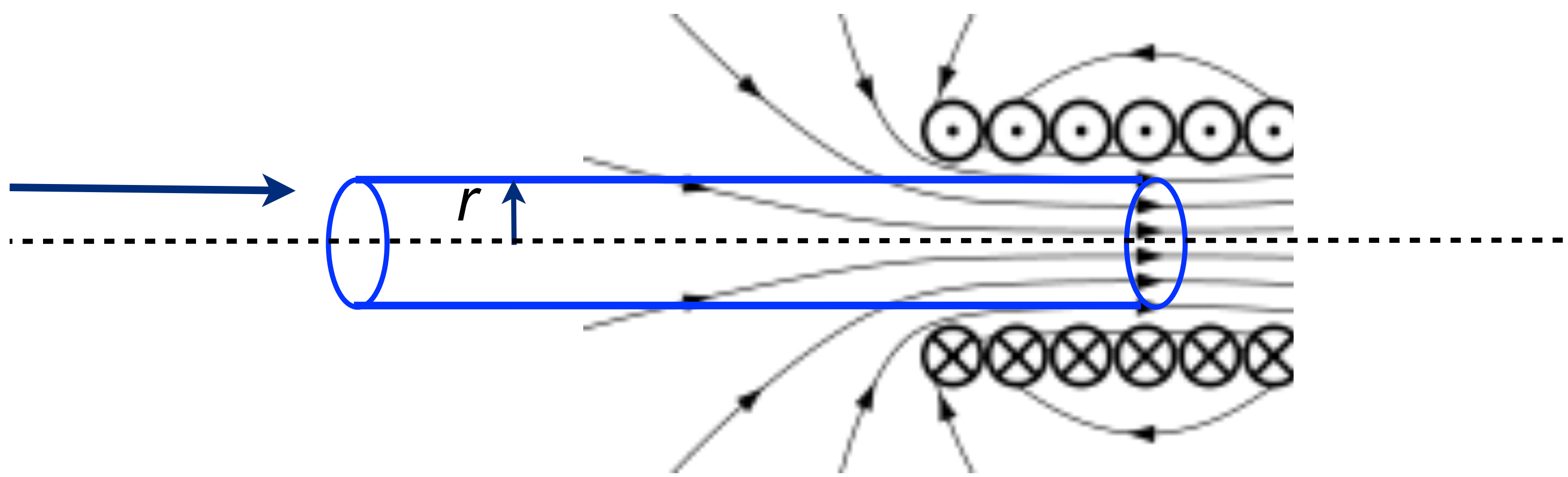
- Imagine particle distribution entering a solenoid magnet, centered on the axis of the magnet, and assume (for now) all trajectories are parallel...



- Treat the “edge” (entrance) of the solenoid as an impulse, but estimate its effect by integrating through the interface

$$\Delta p_\theta \approx q \int_{-\infty}^{0} (\vec{v} \times \vec{B})_\theta dt$$

$$= -\frac{qB_0}{2}\, r$$



Solenoid Focusing

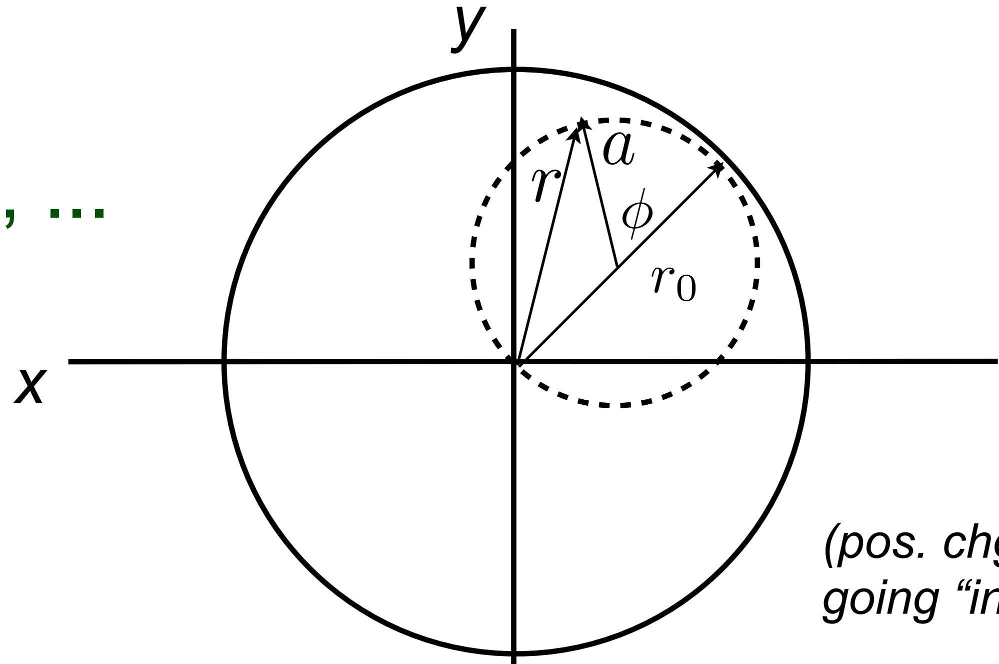
- So, the momentum gained in the theta direction will depend upon it's distance from the solenoid axis, and thus its radius of gyration will be given through

$$\omega = qB_0/m = \frac{v_\perp(r)}{(r/2)} \quad \rightarrow \quad a = r/2$$

- The resulting trajectory will be helical, with radius *a* = $r_0/2$, and the rotation will advance by an amount

$$\phi(s) = \frac{qB_0}{m}\frac{s}{v_\perp}$$

- Thus, ...



$$r = r_0 \cos[\phi(s)/2]$$

(pos. chg. particles going "into" the page)

- Upon exit, assuming the rotation angle is "small", the angular momentum of the beam will be removed.

Solenoid Focusing

- We see that the equation of motion of the particle radius is

$$r'' = -\left(\frac{qB_0}{2mv}\right)^2 r$$

- and hence, solenoid focuses radially
- A "short" solenoid can be interpreted as a "thin lens" of focal length

$$\frac{1}{f} = \left(\frac{qB_0}{2mv}\right)^2 \ell = \left(\frac{B_0}{2(B\rho)}\right)^2 \ell = \frac{\theta_0^2}{\ell} \qquad \text{where } \theta_0 = B_0\ell/2(B\rho)$$

- Note: "thin" lens $\quad \rightarrow \quad \ell/f = \theta_0^2 \ll 1$
- Use solenoids when *Q* is high and momentum is low.
- Note: because the helical rotation is finite, the angular momentum is not entirely removed upon exit. To minimize this effect, can alter the polarities of the solenoids through the (assumed) periodic system.

Solenoids vs. Quadrupoles

- Solenoids often used with lower-energy beams
  - When to change to quadrupoles?
  - No strict answer, but can see there might be a trade-off...

Solenoid:

$$\frac{1}{f} = \left(\frac{B_s}{2\ B\rho}\right)^2 \ell$$

$$\frac{B_s^2}{4(B\rho)^2} = \frac{B_q}{R\ (B\rho)} \longrightarrow B\rho = \frac{R}{4}\ \frac{B_s^2}{B_q}$$

Quadrupole:

$$\frac{1}{f} = \frac{B_q'\ell}{B\rho} = \frac{(B_q/R)\ell}{B\rho}$$

$$B\rho = p/q = A/Q \cdot m_u \gamma \beta c/e$$

- Example: *Bρ* = (50 mm/4) (8 T)^2 / 1T ~ 0.8 T-m
- let *Q/A* = 1/4, *Bρ* = 1 T-m ---> *β* ~ 0.1

Linear Restoring Forces

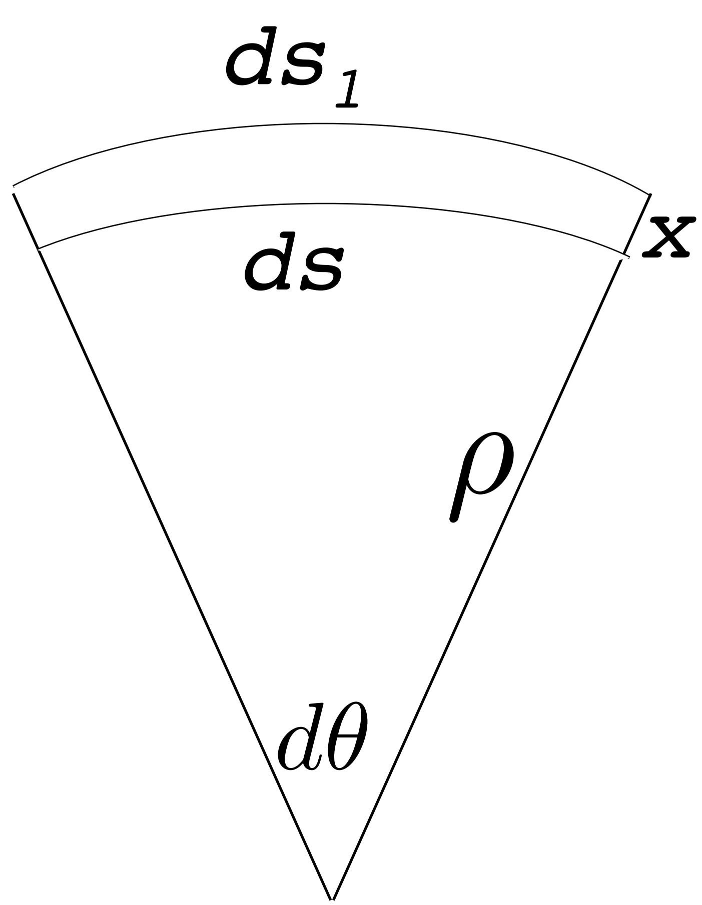
- Wish to look at motion "near" the ideal trajectory of the accelerator system
- Assume linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- Look at radial motion:

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$

$$\frac{ds_1}{\rho + x} = \frac{ds}{\rho}$$



$$\begin{aligned}
\gamma m \frac{d^2(X_d)}{dt^2} &= -e v_s B_0 \\
\gamma m \frac{d^2(X_d + x)}{dt^2} &= -e v_{s1} B_y(X) \\
\gamma m (X_d'' + x'') v_s^2 &= -e v_{s1} B_y(X) \\
\gamma m v_s x'' &= -e \frac{v_{s1}}{v_s} B_y + e B_0 \\
\gamma m v_s x'' &= -e \left[B_y \left(1 + \frac{x}{\rho} \right) - B_0 \right] \\
x'' &= -\frac{e}{p} \left[(B_y - B_0) + B_y \frac{x}{\rho} \right] \\
&\approx -\frac{1}{B\rho} \left[B'x + B_0 \frac{x}{\rho} \right]
\end{aligned}$$

Hill's Equation

- Now, for vertical motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$\begin{aligned}
\gamma m \frac{d^2(Y_d)}{dt^2} &= 0 \\
\gamma m \frac{d^2(Y_d + y)}{dt^2} &= e v_{s1} B_x(Y) \\
\gamma m v_s^2 y'' &= e v_{s1} B_x(Y) \\
\gamma m v_s y'' &= e \frac{v_{s1}}{v_s} B_x \\
\gamma m v_s y'' &= e B_x \left(1 + \frac{x}{\rho}\right) \\
y'' &= \frac{e}{p}\left[B_x(1 + \frac{x}{\rho})\right] \\
&\approx \left(\frac{B'}{B\rho}\right) y
\end{aligned}$$

- So we have,

to lowest order,

$$\boxed{\begin{aligned}
x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right) x &= 0 \\
y'' - \left(\frac{B'}{B\rho}\right) y &= 0
\end{aligned}}$$

Hill's Equation

General Form: $\longrightarrow$ $x'' + K(s)x = 0$

- As accelerate, scale *K* with momentum; becomes purely geometrical

Piecewise Method of Solution

- Hill's Equation: $$x'' + K(s)x = 0$$
  - Though $K(s)$ changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, sector mag, quad, edge, ...)
  - K = 0: $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$
    drift
  - K > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$
    Quad, Gradient Magnet, edge, ...
  - K < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to $s=L$

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K < 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

“Thin Lens” Quadrupole

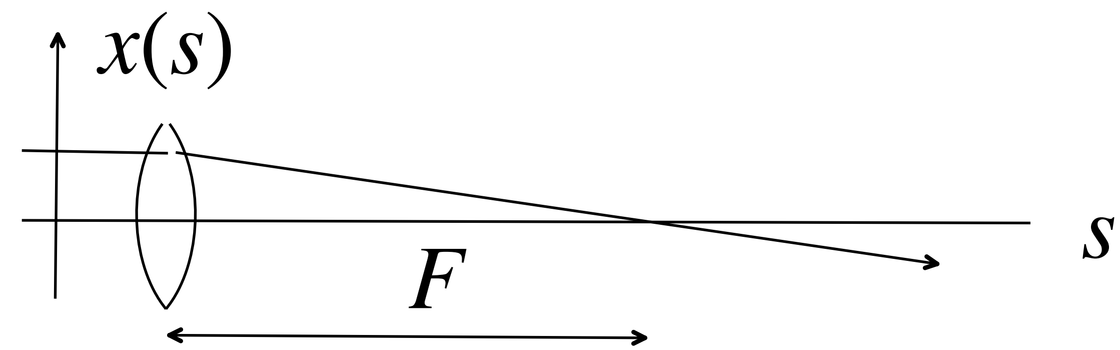
- If quadrupole magnet is short enough, particle’s offset through the quad does not change by much, but the slope of the trajectory does -- acts like a “thin lens” in geometrical optics
- Take limit as L --> 0, while KL remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

  - (similarly, for defocusing quadrupole)

- Valid approx., if F >> L

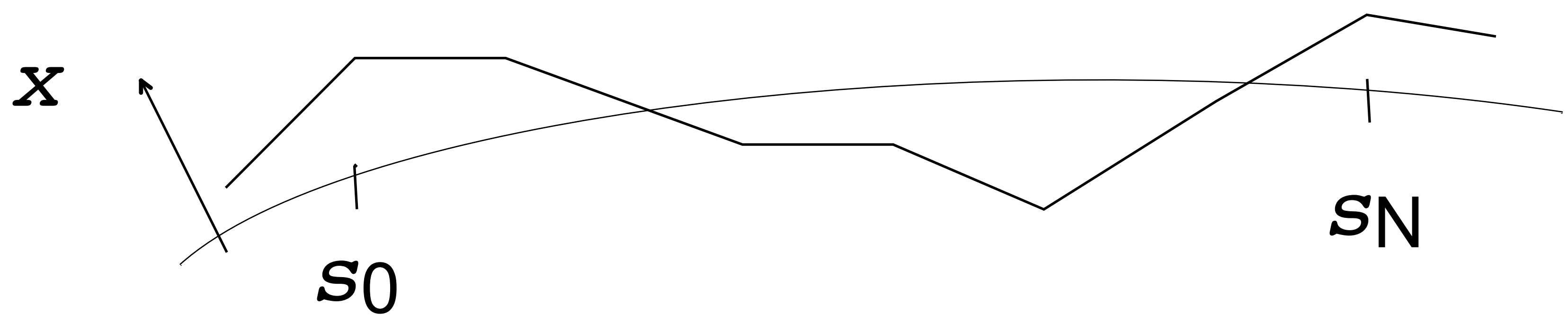
$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$



Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed *via* matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Stability Criterion

- For single pass through a short system of elements, above may be all we need to know to describe the system. But, suppose the "system" is very long and made of many repetitions of the same type of elements (or, perhaps the "repetition" is a complete circular accelerator, for instance) -- how to show that the motion is stable for many (infinite?) passages?
- Look at matrix describing motion for one passage:

$$M = M_N M_{N-1} \cdots M_2 M_1$$

- We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Stability Criterion

v = eigenvector
λ = eigenvalue

$$X_k = M^k X_0 = M^k(AV_1 + BV_2) = A\lambda_1^k V_1 + B\lambda_2^k V_2$$

$$\det M = 1 = \lambda_1\lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ real, gives oscillatory solutions…

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

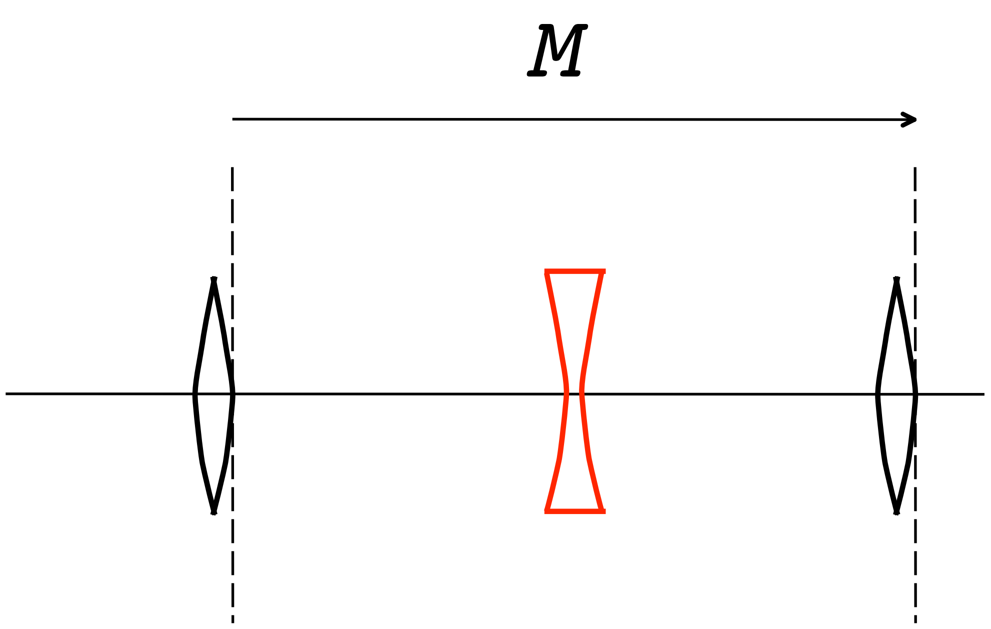
$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda^2 - tr M\lambda + 1 = 0$$
$$\lambda + 1/\lambda = tr M$$
$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = tr M$$

So, μ real (stability)
$\rightarrow |tr M| < 2$

Example: Application to FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix}\begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $tr M = 2 - L^2/F^2$ and thus, for stability,

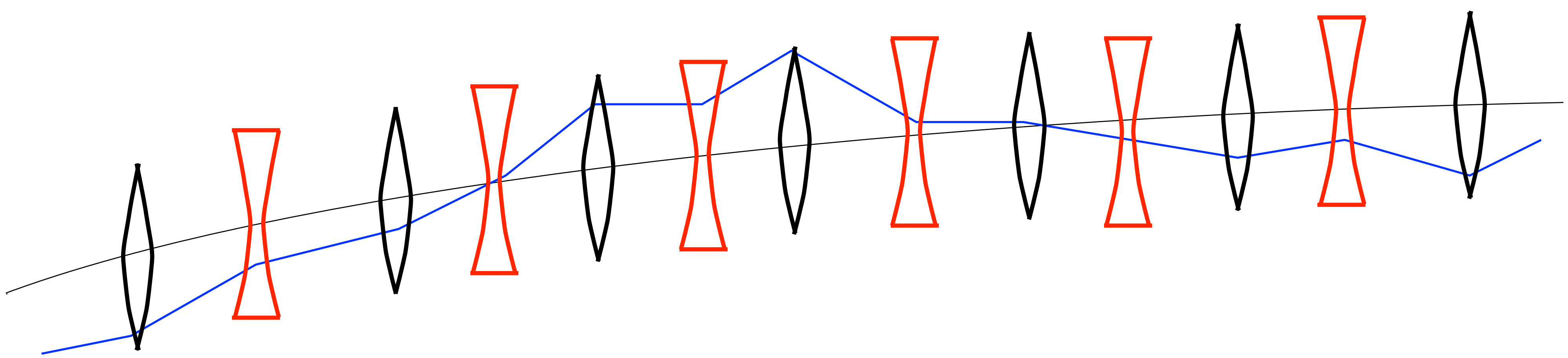
$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

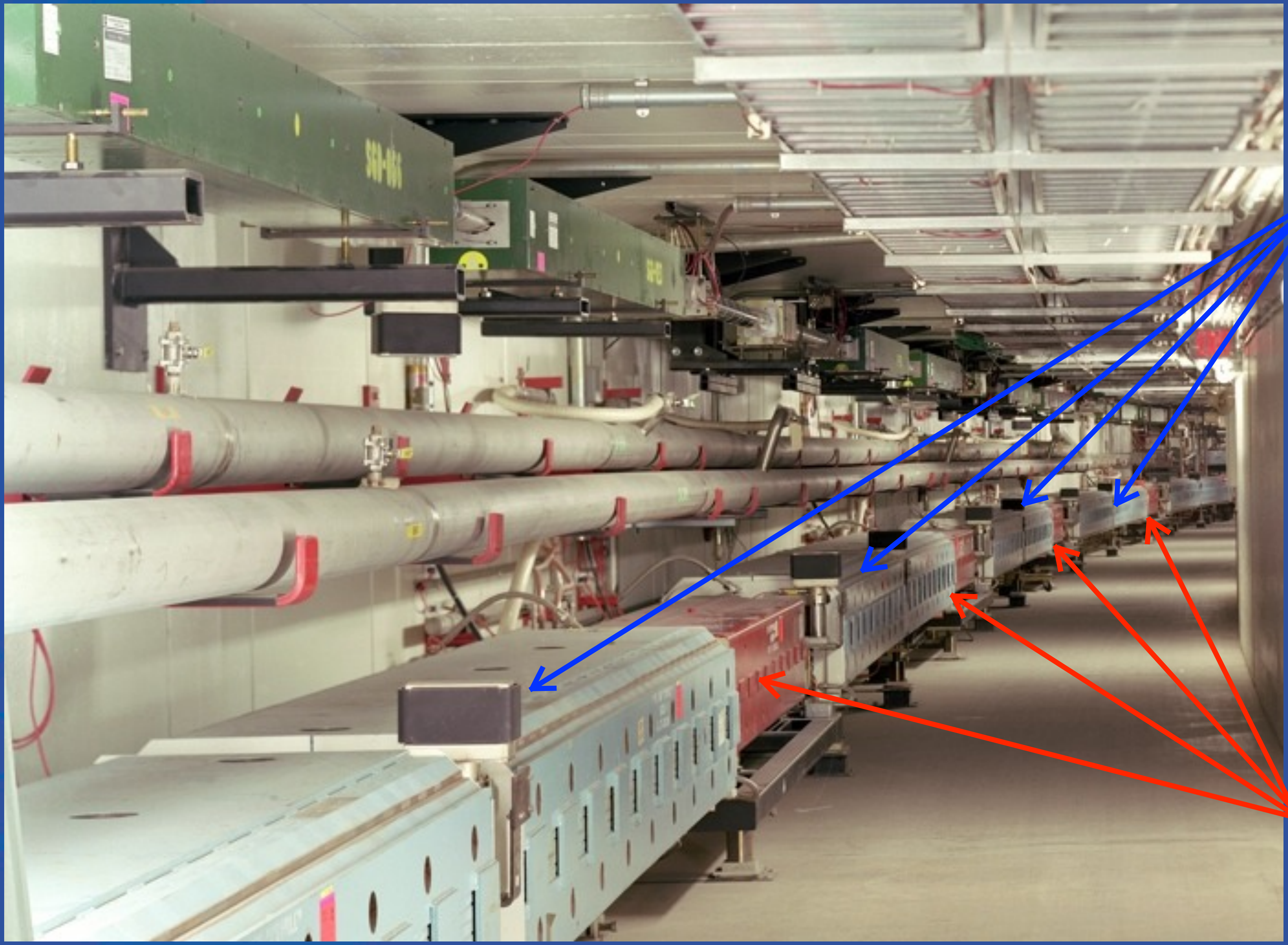
Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size



- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

Example: Fermilab Main Injector



Bending Magnets

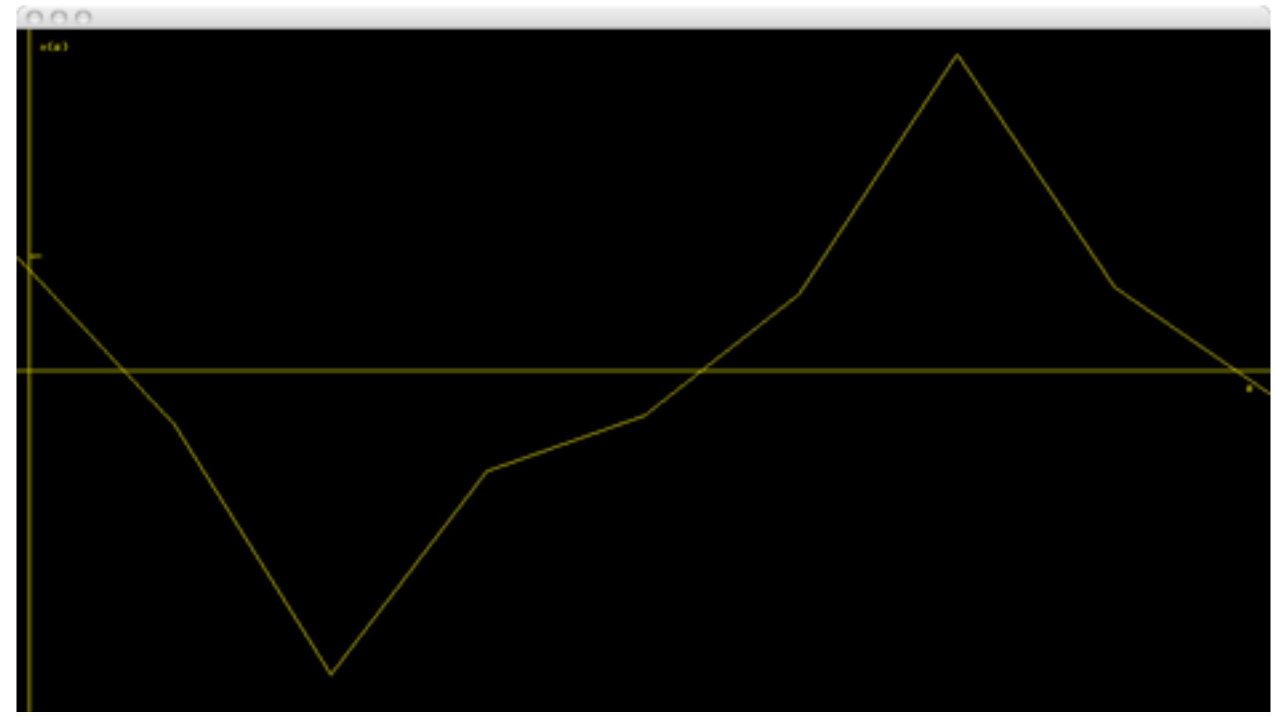
~ 2 mile circumference

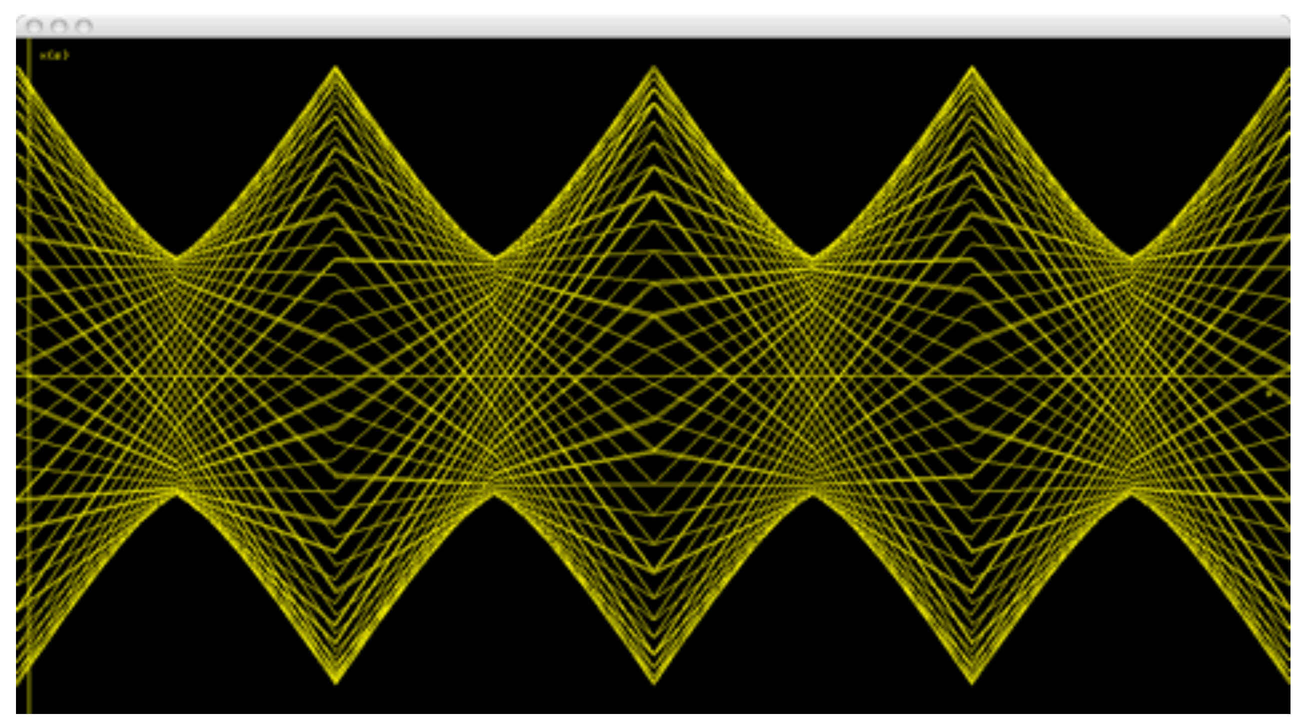
Focusing Magnets

The Notion of an Amplitude Function…

- Track single particle(s) through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

of course! ***coming up…***

Break till Day Two...

- Tomorrow:
  - Analytical formulation of beam optics and transverse oscillations
  - Transverse beam emittance and phase space
  - Chromatic effects, momentum dispersion (and fragment separation)
  - Overview of FRIB design